

Disturbance Attenuating Controller Design for Strict-Feedback Systems with Structurally Unknown Dynamics [★]

Gürdal Arslan ^a and Tamer Başar ^{a,b}

^a *Coordinated Science Laboratory, University of Illinois; 1308 West Main Street; Urbana, IL 61801-2307 / USA; E-mail: (arslan,tbasar)@control.csl.uiuc.edu.*

^b *Author for correspondence.*

**Neural-net based approximators can be used
to design disturbance attenuating adaptive
controllers for strict-feedback systems with
structurally unknown nonlinearities**

Abstract

We consider the problem of robust controller design for a class of single-input single-output nonlinear systems in strict-feedback form with structurally unknown dynamics and also with unknown virtual control coefficients. The unknown nonlinearities in the system dynamics are approximated in terms of a family of basis functions, with the only crucial assumption made being that the parameters that characterize such a neural-network based approximation lie in some known compact sets. In this setup, we design a robust state-feedback controller under which the system output tracks a given signal arbitrarily well, and all signals in the closed-loop system remain bounded. Moreover, a relevant disturbance attenuation inequality is satisfied by the closed-loop signals. We then extend these results to the case where only the output variable is available for feedback. In this case, for tractability, the nonlinear functions in the system dynamics are restricted to depend only on the measured output variable, which results in a strict output-feedback form.

Key words: Adaptive control; Tracking; Neural networks; Nonlinear systems; Robust estimation; State-feedback; Output-feedback.

[★] A shorter version of this paper was presented at IFAC SYSID 2000: Symposium on System Identification, which was held in Santa Barbara, California, during June 21-23, 2000. Original version submitted to Automatica on July 1, 2000, and this revised version on January 15, 2001.

1 Introduction

Control of nonlinear systems for regulation and tracking under various types of uncertainty has been one of the leading research topics of this decade. The types of uncertainty that may exist in a nonlinear system include (but certainly are not limited to) additive exogenous disturbances and lack of knowledge about the nonlinearities. The additive exogenous disturbances can be taken as deterministic or stochastic signals, and depending on this distinction the control problem at hand could be formulated in different settings. Likewise, lack of knowledge about the nonlinearities may be in the form of a finite number of unknown parameters or it may be in a more general form.

When additive uncertainty is of the deterministic type, then one wishes to attenuate its effect on the output, and a viable framework for this is that of nonlinear H^∞ control [26]. Equivalently formulated as a zero-sum differential game, computation of state-feedback controllers in this case boils down to computation of the viscosity solution of a Hamilton-Jacobi-Isaacs (HJI) equation or inequality, which is generally not an easy task. If in addition to the additive uncertainty, there is incomplete knowledge of some or all of the nonlinearities, with the unknowns being a finite number of parameters that describe the unknown nonlinearities, then the problem can also be formulated in the framework of nonlinear H^∞ control, but this time with imperfect state measurements, since the values of the parameters (which constitute additional states) can only be deduced through the regular state measurements [31]. Even though the general nonlinear H^∞ control problem with imperfect state measurements requires infinite-dimensional filters (describing the evolution of the *cost-to-come function* or the *information state*) ([26,18,7]), when the unknown parameters enter the system dynamics linearly the difficulty stemming from infinite dimensionality can be circumvented, as shown in [4] in the context of robust parameter identification. Hence, in this case also, the difficulty in the derivation and computation of robust controllers can be reduced to one of computation of the viscosity solution of a particular HJI equation or inequality. In view of this difficulty, instead of addressing systems with quite general nonlinear dynamics, researchers have started looking into specific (but realistic) structures, with one such structure being provided by “strict feedback” systems. For such structures, and using the tool of *backstepping* [13], it has been possible to solve the HJI equation or inequality iteratively (but explicitly), as documented in [31,9,8], among others. These successes and the fact that a variety of physical systems can be modeled (possibly after a coordinate transformation) in the form of a strict feedback system (see [13,29,27,14,12,19]) drive the formulation of the specific robust tracking problem introduced in Section 3, where within the framework of disturbance driven strict feedback systems, we further relax the assumption of linear parameterization of the nonlinear dynamics, and take the nonlinearities to be quite general, albeit

satisfying certain (not restrictive) conditions regarding their basis function approximator. Derivation of the robust controller involves nonlinear H^∞ identification, and an extended backstepping tool.

In addition to additive uncertainty, if the system also has structural uncertainty (such as some unknown nonlinearities), then we are in the paradigm of robust adaptive control and identification. Most of the successful designs in this context have involved systems where the uncertain part is either linearly parameterized or can be dominated by a linearly parameterized uncertainty, as in [1,17]. For instance, it has been shown in [4] that if the disturbance is persistently exciting, then the parameter estimates, which are optimal in the H^∞ sense, converge to their true values. The identifiers presented in [4] have later been used in [31] to construct adaptive controllers for parametric-strict-feedback nonlinear systems, achieving asymptotic tracking and disturbance attenuation. In the presence of general unknown nonlinearities (that is those of the nonparametric type), the use of neural-network models to identify as well as control a nonlinear system has become popular. The fact that a large class of functions and their derivatives can be approximated on compact sets with arbitrary precision by neural-network models has played a major role in their acceptance; see, for example, [6,10,20]. Driven by these good approximation capabilities, many different identifier and controller designs for nonlinear systems have appeared in the literature; see [23,24,22,11,15,28,30]. In the majority of this previous work, however, with the exception of [24] which essentially studies a first-order system, the issue of robustness with respect to additive disturbances has not been addressed. Rather, in many cases involving a neural network based controller, the system model has been assumed to be noise-free, with the main difficulty to overcome in this case appearing to be the effect of the approximation error on the closed-loop system. Very often, some measures need to be taken to prevent the states from leaving a compact set in which the approximation error is small. In the recent work [16], for example, the unknown nonlinear functions propagating through the higher steps in the backstepping design are assumed to be bounded by a known function multiplied by an unknown scalar. Among different approaches, the assumption that the optimal parameters that characterize a basis-function approximation of the unknown nonlinearities lie within a known compact set, as in [24], seems to be more realistic, and hence will be employed in this work.

Accordingly, we consider in this paper the class of problems where the unknown nonlinearities in the system description are not parameterized, and the system dynamics are subject to additive unknown disturbances. This constitutes a generalization of the results of [3], [31], and [5] to the case where the nonlinearities are more general. We should point out that if the nonlinearities can be expressed as unknown linear combinations of some known basis functions for which the resulting approximation errors are square integrable, then the approximation errors can be absorbed into the external disturbances

and the results of [31] and [5] would then apply. However, if the nonlinearities are unknown, as in this paper, we cannot guarantee that the approximation errors are square integrable. Moreover, if a reference signal is prescribed to satisfy an appropriate persistency of excitation condition, rather than solely relying on the external disturbances, then the approximation errors generally cannot be square integrable, in which case one must account for their effects. Hence, in this work we take the unknown nonlinearities in the plant dynamics to be quite general, and for this general class we show that the output tracking error can be made as small as desired in a prespecified amount of time, while keeping all other signals bounded. Also, the closed-loop signals satisfy a disturbance attenuation inequality with respect to an equivalent disturbance term, which is composed of the external disturbances and the approximation errors. As a result of the satisfaction of this inequality, the tracking error converges to zero whenever the approximation errors are zero and the external disturbances are square-integrable. Furthermore, if a relevant persistency of excitation condition is satisfied and the unknown nonlinearities admit perfect expansions in terms of their respective basis functions, then the parameter estimates converge to the optimal values for these parameters, provided that the external disturbances are square-integrable. To achieve this, we use modified versions of the identifiers developed in [31], and employ the backstepping design methodology, when the state variables are available for feedback. In the output-feedback case, however, we adopt the estimators introduced in [5] to estimate the state variables and the unknown optimal parameters. Then applying the backstepping design on the estimates of the unmeasured states yields the desired controller.

In the sequel, we first introduce some notation that is used throughout the paper; see Section 2. Then, in Section 3, we define our objectives in more precise terms. Following this, we introduce, in Section 4, the identifiers used in the paper for the state-feedback case. In Section 5, we present the controller design steps with full state and derivative measurements, which are used later in Section 6 in the study of the more realistic case where only the state variables are available. The issue of persistency of excitation, which is related to parameter convergence, is addressed in Section 7. Extensions of the results presented in Sections 5 and 6 to the output-feedback case are discussed in Section 8. In Section 9, a numerical example is presented to demonstrate the effectiveness of the designed controller for the state-feedback case. Finally, the paper ends with the concluding remarks of Section 10.

2 Notation

The following notation is used throughout the paper, unless stated otherwise.

- For any positive integer m , \mathcal{R}^m denotes the m dimensional Euclidean space.
- $|\cdot|$ denotes the standard Euclidean norm, and $|\cdot|_Q$ the weighted norm, that is $|x|_Q^2 = x^T Q x$.
- $(\cdot)^T$ denotes the transpose of a vector or a matrix.
- \mathcal{L}_2 denotes the space of Lebesgue measurable and square integrable functions defined on $[0, \infty)$.
- For a vector-valued function $f(t)$, $\|f\| := \sup_{t \geq 0} |f(t)|$.
- $x_{[i]} := [x_1^T, \dots, x_i^T]^T$, where x_j , $j \in \{1, \dots, i\}$, can be a scalar or a vector.
- $\lambda_{\min}(\cdot)$ denotes the minimum eigenvalue of a symmetric matrix.
- I denotes the identity matrix of appropriate dimension.
- A continuous function $f(x) : [0, \infty) \mapsto [0, \infty)$ is said to be class \mathcal{K}_∞ if $f(0) = 0$, f is increasing and $\lim_{x \uparrow \infty} f(x) = \infty$.
- For a function $f(t)$, $D^+ f(t) := \limsup_{h \downarrow 0} \frac{f(t+h) - f(t)}{h}$.
- If $g : \mathcal{R} \mapsto \mathcal{R}$, $g(x) = O(x)$ means that $\limsup_{x \rightarrow 0} \left| \frac{g(x)}{x} \right|$ is finite.

3 Problem Formulation

We consider a general nonlinear strict-feedback system described by the following equations:

$$\begin{aligned}
 \dot{x}_i &= b_i(x^i) x_{i+1} + g_i(x^i) + h_i^T(x^i) w_i, & x^i &:= x_{[i]}, & i &= 1, \dots, r-1 \\
 \dot{x}_r &= b_r(x^r) u + g_r(x^r) + h_r^T(x^r) w_r, & x^r &:= [x_r^T, x_z^T]^T \\
 \dot{x}_z &= g_z(x^r, w_z) \\
 y &= x_1,
 \end{aligned} \tag{1}$$

where $x := x^r \in \mathcal{R}^n$ is the state vector¹; $x_z \in \mathcal{R}^{n-r}$ represents the state of the zero dynamics; $u \in \mathcal{R}$ is the control input; y is the scalar output; and w_i , $i = 1, \dots, r$, and w_z are vector-valued external disturbances. The nonlinear functions h_i , b_i , g_i , $i = 1, \dots, r$, and g_z are sufficiently smooth, with only h_i 's known. The objective is to force y track a prespecified signal $y_d(t)$ by picking the control input u appropriately, to be made precise later in Sections 5 and 6. To accomplish this objective, the unknown nonlinear terms g_i and b_i will be approximated in terms of a set of sufficiently smooth basis functions, ϕ_i , and φ_i , $i = 1, \dots, r$. In particular, these basis functions can be radial basis functions of an appropriate neural network, which possess the property that an appropriate linear combination of them can approximate a continuous function arbitrarily well on any compact set; see Theorem 5 in [6] for details. The

¹ We have intentionally introduced a ‘‘discontinuity’’ in the definition of x^i in going from $i = r-1$ to r , in order to simplify the notation in the development of Sections 5 and 6.

optimal parameters used in such an approximation are defined as: for $i = 1, \dots, r$,

$$\begin{aligned}\theta_i &:= \arg \min_{\theta_i \in \mathcal{R}^{d_i}} \max_{x^i \in B_i} |g_i(x^i) - \phi_i^T(x^i)\check{\theta}_i| \\ \vartheta_i &:= \arg \min_{\vartheta_i \in \mathcal{R}^{m_i}} \max_{x^i \in B_i} |b_i(x^i) - \varphi_i^T(x^i)\check{\vartheta}_i|\end{aligned}$$

where B_i are some sufficiently large compact subsets of appropriate Euclidean spaces. In other words, if the functions g_i and b_i were known, and we wanted to use only an approximation for them in terms of the given basis functions, the coefficients of the approximants for g_i and b_i would be the components of θ_i and ϑ_i , respectively. The approximation errors are defined as: for $i = 1, \dots, r$,

$$e_i^g(x^i) := g_i(x^i) - \phi_i^T(x^i)\theta_i, \text{ and } e_i^b(x^i) := b_i(x^i) - \varphi_i^T(x^i)\vartheta_i. \quad (2)$$

Note that the functions $e_i^g(x^i)$ and $e_i^b(x^i)$ are continuous, and hence they are uniformly bounded on any compact set. We now make the following assumptions, which will be used throughout the paper.

Assumption 1 $b_i(x^i) > k_b$, for some $k_b > 0$, $\forall x^i \in B_i$, $i = 1, \dots, r$.

Assumption 2 $|h_i(x^i)|^2 > k_h$, for some $k_h > 0$, $\forall x^i \in B_i$, $i = 1, \dots, r$.

Assumption 3 *There exists a known sufficiently smooth, strictly convex, radially unbounded function $P_i : \mathcal{R}^{m_i+d_i} \mapsto \mathcal{R}$ such that $P_i(\xi_i) \leq 0$, where $\xi_i := [\vartheta_i^T, \theta_i^T]^T$. It is further assumed that if $P_i(\bar{\xi}_i) \leq 1$ for any $\bar{\xi}_i \in \mathcal{R}^{m_i+d_i}$, then $\varphi_i^T(x^i)\bar{\vartheta}_i \geq k$, $\forall x^i \in B_i$, where $\bar{\vartheta}_i$ is the vector composed of the first m_i components of $\bar{\xi}_i$, $i = 1, \dots, r$, and k is a known positive constant.*

Assumption 4 *The zero subsystem is input-to-state stable with respect to $x_{[r]}$ and w_z . This requires the existence of a class \mathcal{K}_∞ function ψ such that*

$$\sup_{s \in [0, t]} |x_z(s)| \leq \psi \left(|x_z(0)| + \sup_{s \in [0, t]} \sqrt{|x_{[r]}(s)|^2 + |w_z(s)|^2} \right), \quad \forall t \geq 0.$$

See [25] for the notion of input-to-state stability.

Assumption 5 $w_1(t), \dots, w_r(t)$, and $w_z(t)$ are Lebesgue measurable functions. Also, $\max_{i \in \{1, \dots, r\}} \|w_i\| \leq M_w$, and $\|w_z\| \leq M_w$, for some known $M_w > 0$.

Assumption 6 *The reference signal $y_d(t)$ is r times continuously differentiable, where $y_d^{(0)} := y_d$ and its derivatives $y_d^{(1)}, \dots, y_d^{(r)}$ are known. Furthermore, $|y_d^{[r]}| \leq M_d$, for some $M_d > 0$, where $y_d^{[i]} := [y_d^{(0)}, y_d^{(1)}, \dots, y_d^{(i)}]^T$, $i = 1, \dots, r$.*

Assumptions 1 and 2 are made to avoid singularities in control and identification, respectively. Assumption 3 is a generalization of the standard assumption

(made in the robust adaptive control literature) that the unknown parameters are bounded. If, for instance, the unknown parameter ξ_i is known to satisfy $|\xi_i| \leq N_i$, for some constant $N_i > 0$, then the function P_i can be taken as: $P_i(\xi_i) = |\xi_i|^2 - N_i^2$. But, Assumption 3 is much more general than this standard assumption, and is necessary to keep the estimates of the unknown parameters bounded, as well as to keep the estimates of the unknown virtual control coefficients bounded away from zero; see Section 9 for some examples. The boundedness of the zero dynamics state $x_z(t)$, when the remaining states are bounded, is guaranteed by Assumption 4. Finally, Assumption 5 is required to maintain the boundedness of the overall system, whereas Assumption 6 is quite standard, and is made to be able to carry out the backstepping design.

4 The Identifiers

Since the parameters θ_i and ϑ_i are unknown, they have to be identified by using the online information. For this purpose, the following modified versions of the identifiers introduced in [31] are adopted: for $i = 1, \dots, r$,

Identifier 1:

$$\dot{\hat{\xi}}_i = \text{Proj}[\tau_i, P_i(\hat{\xi}_i), \Sigma_i], \quad \tau_i := \Sigma_i \Phi_i (\dot{x}_i - \Phi_i^T \hat{\xi}_i) / |h_i|^2 \quad (3)$$

Identifier 2:

$$\begin{aligned} \dot{\hat{\xi}}_i &= \text{Proj}[\tau_i, P_i(\hat{\xi}_i), \Sigma_i], \quad \tau_i := \Sigma_i \Phi_i (x_i - \hat{x}_i) / (\epsilon |h_i|), \text{ for some } \epsilon > 0 \\ \dot{\hat{x}}_i &= \Phi_i^T \hat{\xi}_i + |h_i| (x_i - \hat{x}_i) / \epsilon, \quad \hat{x}_i(0) = x_i(0) \end{aligned}$$

where the initial condition $\hat{\xi}_i(0)$ is such that $P_i(\hat{\xi}_i(0)) \leq 0$, and

$$\begin{aligned} \Phi_i &:= [\varphi_i^T x_{i+1}, \phi_i^T]^T, \quad \hat{\xi}_i := [\hat{\vartheta}_i^T, \hat{\theta}_i^T]^T, \quad x_{r+1} := u \\ \dot{\Sigma}_i &= (\eta - 1) \Sigma_i \Phi_i \Phi_i^T \Sigma_i / |h_i|^2, \quad \Sigma_i(0) = I, \quad \text{for some } 0 < \eta \leq 1. \end{aligned} \quad (4)$$

The projection function is defined by $\text{Proj}[\tau, P(\hat{\xi}), \Gamma] := \mathcal{P}[\tau, P(\hat{\xi}), \Gamma]\tau$, where

$$\mathcal{P}[\tau, P(\hat{\xi}), \Gamma] := \begin{cases} I & \text{if } P < 0 \text{ or } (\partial P / \partial \hat{\xi})^T \tau \leq 0 \\ I - \frac{P \Gamma (\partial P / \partial \hat{\xi}) (\partial P / \partial \hat{\xi})^T}{(\partial P / \partial \hat{\xi})^T \Gamma (\partial P / \partial \hat{\xi})} & \text{if } P \geq 0 \text{ and } (\partial P / \partial \hat{\xi})^T \tau > 0 \end{cases}.$$

It is locally Lipschitz in its arguments, and is utilized in (3) and (4) to guarantee $\|P_i(\hat{\xi}_i(t))\| \leq 1$, $i = 1, \dots, r$; see [21] for the properties of the projection

function. As a result, the parameter estimates generated by Identifiers 1 and 2 are always uniformly bounded, and they always satisfy $\varphi_i^T(x^i)\hat{\vartheta}_i \geq k, \forall x^i \in B_i, i = 1, \dots, r$. Another observation is that the covariance matrices $\Sigma_i(t)$ generated by (4) are nonnegative definite and nonincreasing in the matrix sense, which implies: $0 \leq \Sigma_i(t) \leq I, \forall t \geq 0, i = 1, \dots, r$.

The reason for the use of Identifiers 1 and 2 introduced above, instead of Lyapunov based estimators commonly seen in the literature, is that Identifiers 1 and 2 are robust with respect to the external disturbances; see [31]. This robustness property is essential in establishing a disturbance attenuation property for the overall system, as will be clear later. Furthermore, in the absence of approximation errors, the parameter estimates generated by Identifiers 1 and 2 converge to their optimal values provided that the external disturbances are square integrable and an appropriate persistency of excitation condition is satisfied. Finally, we note that Identifier 1 requires the knowledge of the state variables as well as their derivatives, whereas Identifier 2 requires the knowledge of the state variables only. In the next two sections, we construct adaptive controllers, which use the parameter estimates generated by (3) or (4) depending on the available information.

5 Full-State and Derivative Measurements

In the development of this section, we assume that in addition to the state variables $x_1(t), \dots, x_r(t), x_z(t)$, also the derivatives $\dot{x}_1(t), \dots, \dot{x}_r(t)$, are available. This is compatible with the information requirement of Identifier 1, and hence the parameter estimates generated by Identifier 1 will be used to construct an adaptive controller that forces the output $y(t)$ of the system (1) track the reference signal $y_d(t)$. Before proceeding with the design steps, the dynamics of Identifier 1 are rewritten in the following more convenient form: for $i = 1, \dots, r$,

$$\begin{aligned} \dot{\hat{\xi}}_i &= \mathcal{P}_i \Sigma_i \Phi_i h_i^T v_i / |h_i|^2, \quad \mathcal{P}_i := \mathcal{P}[\tau_i, P_i(\hat{\xi}_i), \Sigma_i] \\ v_i &:= w_i + h_i((\varphi_i^T \tilde{\vartheta}_i + e_i^b)x_{i+1} + \phi_i^T \tilde{\theta}_i + e_i^g) / |h_i|^2 \\ \tilde{\vartheta}_i &:= \vartheta_i - \hat{\vartheta}_i, \quad \tilde{\theta}_i := \theta_i - \hat{\theta}_i. \end{aligned}$$

Step 1: The output tracking error is defined as: $z_1 := x_1 - y_d$. It follows that:

$$\dot{z}_1 = b_1 x_2 + g_1 + h_1^T w_1 - y_d^{(1)}. \quad (5)$$

Viewing x_2 as a virtual control input, we pick its desired value as α_1 below, to drive z_1 to zero:

$$\alpha_1 := \bar{\alpha}_1 - \beta_1 z_1 / (\varphi_1^T \hat{\vartheta}_1), \quad \bar{\alpha}_1 := [y_d^{(1)} - \phi_1^T \hat{\theta}_1 - |h_1|^2 z_1 / (2\gamma_1^2)] / (\varphi_1^T \hat{\vartheta}_1)$$

where $\beta_1(z_1, y_d^{[1]}, \hat{\xi}_1) > 0$ is a sufficiently smooth design function, and $\gamma_1 > 0$ is an arbitrary scalar. However, since x_2 is not the actual control input, the objective of driving z_1 to zero this way cannot be accomplished. We then introduce the error term z_2 as: $z_2 := x_2 - \alpha_1(z_1, y_d^{[1]}, \hat{\xi}_1)$, and rewrite (5) as:

$$\dot{z}_1 = \varphi_1^T \hat{\vartheta}_1 z_2 - \beta_1 z_1 + h_1^T v_1 - |h_1|^2 z_1 / (2\gamma_1^2).$$

Step k ($1 < k < r$): Assume the following structure from the $(k-1)$ -th step:

$$\begin{aligned} z_{i+1} &= x_{i+1} - \alpha_i(z_{[i]}, y_d^{[i]}, \hat{\xi}_{[i]}) \\ \dot{z}_i &= \varphi_i^T \hat{\vartheta}_i z_{i+1} - \beta_i z_i - \varphi_{i-1}^T \hat{\vartheta}_{i-1} z_{i-1} + h_i^T v_i + \sum_{j=1}^{i-1} [p_{ij}^T + q_{ij}^T] v_j \\ &\quad - |h_i|^2 z_i / (2\gamma_i^2) - \sum_{j=1}^{i-1} [|p_{ij}|^2 + \bar{q}_{ij}^2] z_i / (2\gamma_j^2) \\ \varphi_0 &= \hat{\vartheta}_0 = z_0 = 0, \end{aligned}$$

where α_i is the desired value of x_{i+1} , $\beta_i(z_{[i]}, y_d^{[i]}, \hat{\xi}_{[i]}) > 0$ is a sufficiently smooth design function, $\gamma_i > 0$ are some arbitrary constants, $i = 1, \dots, k-1$. The functions $p_{ij}(z_{[i-1]}, y_d^{[i-1]}, \hat{\xi}_{[i-1]})$, $\bar{q}_{ij}(z_{[i]}, y_d^{[i-1]}, \hat{\xi}_{[i-1]})$ are smooth, q_{ij} is continuous, and the latter two satisfy $\bar{q}_{ij}^2 \geq |q_{ij}|^2$, $j = 1, \dots, i-1$, $i = 2, \dots, k-1$. Using the induction hypothesis above, we compute the derivative of z_k as:

$$\dot{z}_k = b_k x_{k+1} + g_k + h_k^T w_k + r_k + \sum_{j=1}^{k-1} [p_{kj}^T + q_{kj}^T] v_j$$

where

$$\begin{aligned} r_k &:= - \sum_{i=1}^{k-1} \frac{\partial \alpha_{k-1}}{\partial z_i} \left[\varphi_i^T \hat{\vartheta}_i z_{i+1} - \beta_i z_i - \varphi_{i-1}^T \hat{\vartheta}_{i-1} z_{i-1} - |h_i|^2 z_i / (2\gamma_i^2) \right] \\ &\quad + \sum_{i=1}^{k-1} \frac{\partial \alpha_{k-1}}{\partial z_i} \sum_{j=1}^{i-1} [|p_{ij}|^2 + \bar{q}_{ij}^2] z_i / (2\gamma_j^2) - \sum_{i=1}^k \frac{\partial \alpha_{k-1}}{\partial y_d^{(i-1)}} y_d^{(i)} \\ p_{kj} &:= - \frac{\partial \alpha_{k-1}}{\partial z_j} h_j - \sum_{m=j+1}^{k-1} \frac{\partial \alpha_{k-1}}{\partial z_m} p_{mj}, \quad j = 1, \dots, k-1 \\ q_{kj} &:= - \sum_{m=j+1}^{k-1} \frac{\partial \alpha_{k-1}}{\partial z_m} q_{mj} - \frac{h_j \Phi_j^T \Sigma_j \mathcal{P}_j^T}{|h_j|^2} \frac{\partial \alpha_{k-1}}{\partial \hat{\xi}_j}, \quad j = 1, \dots, k-1. \end{aligned}$$

Now viewing x_{k+1} as the virtual control input at this step, its desired value is picked as:

$$\alpha_k := \bar{\alpha}_k - \beta_k z_k / (\varphi_k^T \hat{\vartheta}_k), \quad (6)$$

where

$$\begin{aligned} \bar{\alpha}_k &:= \frac{-r_k - \varphi_{k-1}^T \hat{\vartheta}_{k-1} z_{k-1} - \phi_k^T \hat{\theta}_k}{\varphi_k^T \hat{\vartheta}_k} - \frac{|h_k|^2 z_k}{2\gamma_k^2 (\varphi_k^T \hat{\vartheta}_k)} - \sum_{j=1}^{k-1} \frac{|p_{kj}|^2 + \bar{q}_{kj}^2}{2\gamma_j^2 (\varphi_k^T \hat{\vartheta}_k)} z_k \\ \bar{q}_{kj}^2 &:= (k-j) \left[\sum_{m=j+1}^{k-1} \left| \frac{\partial \alpha_{k-1}}{\partial z_m} \right|^2 \bar{q}_{mj}^2 + 4 \frac{|\Phi_j|^2}{|h_j|^2} \left| \frac{\partial \alpha_{k-1}}{\partial \hat{\xi}_j} \right|^2 \right] \quad j = 1, \dots, k-1, \end{aligned}$$

$\beta_k(z_{[k]}, y_d^{[k]}, \hat{\xi}_{[k]}) > 0$ is a sufficiently smooth design function, and $\gamma_k > 0$ is an arbitrary scalar. Since x_{k+1} is not the actual control input, the error term z_{k+1} is defined as: $z_{k+1} := x_{k+1} - \alpha_k(z_{[k]}, y_d^{[k]}, \hat{\xi}_{[k]})$, and \dot{z}_k is rewritten as:

$$\begin{aligned} \dot{z}_k &= \varphi_k^T \hat{\vartheta}_k z_{k+1} - \beta_k z_k - \varphi_{k-1}^T \hat{\vartheta}_{k-1} z_{k-1} + h_k^T v_k + \sum_{j=1}^{k-1} [p_{kj}^T + q_{kj}^T] v_j \\ &\quad - |h_k|^2 z_k / (2\gamma_k^2) - \sum_{j=1}^{k-1} [|p_{kj}|^2 + \bar{q}_{kj}^2] z_k / (2\gamma_j^2). \end{aligned}$$

Since all of the relevant definitions and results of Step k are consistent with the induction hypothesis, we conclude that the induction hypothesis holds true for $k = 2, \dots, r-1$.

Step r : It is easy to see that the results of Step k hold also for $k = r$, with $x_{r+1} := u$, and $z_{r+1} = 0$. Since u is the actual control input, z_{r+1} can be made zero by picking the control input as: $u = \alpha_r$, where α_r is obtained by setting $k = r$ in (6).

Having completed the controller design, we now introduce the functions: $V_i(t) = z_i^2(t)/2$, $i = 1, \dots, r$, whose time-derivatives satisfy the inequalities:

$$\dot{V}_i \leq \varphi_i^T \hat{\vartheta}_i z_i z_{i+1} - \beta_i z_i^2 - \varphi_{i-1}^T \hat{\vartheta}_{i-1} z_{i-1} z_i + \sum_{j=1}^{i-1} \gamma_j^2 |v_j|^2 + \gamma_i^2 |v_i|^2 / 2, \quad \text{or} \quad (7)$$

$$\dot{V}_i \leq \varphi_i^T \hat{\vartheta}_i z_i z_{i+1} - \frac{b_i}{\varphi_i^T \hat{\vartheta}_i} \beta_i z_i^2 - \varphi_{i-1}^T \hat{\vartheta}_{i-1} z_{i-1} z_i + \gamma_i^2 |\bar{v}_i|^2 / 2 + \sum_{j=1}^{i-1} \gamma_j^2 |v_j|^2 \quad (8)$$

where $\bar{v}_i := w_i + h_i((\varphi_i^T \tilde{\vartheta}_i + e_i^b)(z_{i+1} + \bar{\alpha}_i) + \phi_i^T \tilde{\theta}_i + e_i^g) / |h_i|^2$.

If the virtual control input x_i were the actual control input, then we would

be able to make the error term z_i exactly equal to zero by picking $x_i = \alpha_{i-1}$, $i \in \{2, \dots, r\}$. Since this is not the case, we fix some arbitrary upper bounds M_i on the error terms z_i , and note that if the design functions β_i are sufficiently large (picked depending on M_i), and the error terms z_i initially satisfy their specified bounds M_i , then we always have $|z_i(t)| \leq M_i$. This observation is made precise in the following remark:

Remark 7 *Let $M_i > 0$ be some arbitrary upper bound on $|z_i|$, $i = 1, \dots, r$ (with $M_0 = M_{r+1} = 0$ for notational consistency). Assume that the design functions $\beta_i(z_{[i]}, y_d^{[i]}, \hat{\xi}_{[i]})$ are picked recursively, $i = 1, \dots, r-1$, such that they satisfy*

$$\beta_i > \max_{\bar{B}_i} \frac{|\varphi_i^T \hat{\vartheta}_i|}{k_b M_i^2} \left[|\varphi_i^T \hat{\vartheta}_i| M_i M_{i+1} + |\varphi_{i-1}^T \hat{\vartheta}_{i-1}| M_{i-1} M_i + \frac{\gamma_i^2 |\bar{v}_i|^2}{2} + \sum_{j=1}^{i-1} \gamma_j^2 M_{v_j}^2 \right]$$

where

$$\bar{B}_i := \left\{ (z_{[i+1]}, y_d^{[i]}, \hat{\xi}_{[i]}, w_{[i]}) : \begin{array}{l} |z_j| \leq M_j, j = 1, \dots, i+1, |y_d^{[i]}| \leq M_d, \\ P_m(\hat{\xi}_m) \leq 1, |w_m| \leq M_w, m = 1, \dots, i \end{array} \right\}$$

$$M_{v_i} := \max_{\bar{B}_i} |w_i + h_i((\varphi_i^T \tilde{\vartheta}_i + e_i^b)(z_{i+1} + \alpha_i) + \phi_i^T \tilde{\theta}_i + e_i^g) / |h_i|^2|.$$

Let M_{z_0} be an arbitrary upper bound on the initial condition $|x_z(0)|$, and M_z be the corresponding upper bound on $|x_z|$. From Assumption 4, M_z is defined as:

$$M_z := \psi \left(M_{z_0} + \max_{\bar{B}_{r-1}} \sqrt{\sum_{m=1}^r (z_m + \alpha_{m-1})^2 + M_w^2} \right)$$

where $\alpha_0 := y_d$. Finally assume that the last design function β_r is picked such that it satisfies

$$\beta_r > \max_{\bar{B}_r} \frac{|\varphi_r^T \hat{\vartheta}_r|}{k_b M_r^2} \left[|\varphi_{r-1}^T \hat{\vartheta}_{r-1}| M_{r-1} M_r + \frac{\gamma_r^2 |\bar{v}_r|^2}{2} + \sum_{j=1}^{r-1} \gamma_j^2 M_{v_j}^2 \right]$$

where

$$\bar{B}_r := \left\{ (z_{[r]}, y_d^{[r]}, \hat{\xi}_{[r]}, w_{[r]}, x_z) : \begin{array}{l} |z_j| \leq M_j, P_j(\hat{\xi}_j) \leq 1, |w_j| \leq M_w, \\ j = 1, \dots, r, |y_d^{[r]}| \leq M_d, |x_z| \leq M_z \end{array} \right\}$$

$$M_{v_r} := \max_{\bar{B}_r} |w_r + h_r((\varphi_r^T \tilde{\vartheta}_r + e_r^b)\alpha_r + \phi_r^T \tilde{\theta}_r + e_r^g) / |h_r|^2|.$$

If $|z_i(0)| \leq M_i$, $i = 1, \dots, r$, $|x_z(0)| \leq M_{z_0}$, and the design functions β_i satisfy the lower bounds delineated above, then the upper right derivative of

$$V(t) := \max \left\{ \frac{z_1^2(t)}{M_1^2}, \dots, \frac{z_r^2(t)}{M_r^2} \right\}$$

has the property:

$$V(t) = 1 \Rightarrow D^+V(t) < 0, \quad \forall t \geq 0.$$

This observation, which essentially establishes the boundedness of the error terms z_i provided that the design functions are picked appropriately, now leads to the following theorem, where the boundedness of all closed-loop signals as well as the tracking, disturbance attenuation, and parameter convergence results are stated and proven.

Theorem 8 Consider the closed-loop system described by (1), (3), and the designed controller, under Assumptions 1,...,6. Assume that $|z_i(0)| \leq M_i$, $|x_z(0)| \leq M_{z_0}$, and the design functions β_i satisfy the lower bounds specified in Remark 7. Then:

- (1) $\|z_i(t)\| \leq M_i$, $\|P_i(\hat{\xi}_i(t))\| \leq 1$, $i = 1, \dots, r$, $\|x_z(t)\| \leq M_z$.
- (2) Given an arbitrary finite time $\bar{T} > 0$, and an arbitrary small number $\rho > 0$, which represents the maximum tracking error after the time \bar{T} , such that $\rho \leq M_1$, if

$$\beta_1 > \frac{M_1^2 - \rho^2 + 2 \max_{\bar{B}_1} (|\varphi_1^T \hat{\vartheta}_1| M_1 M_2 + \gamma_1^2 |\bar{v}_1|^2 / 2) \bar{T}}{2\rho^2 \bar{T} k_b / \max_{\bar{B}_1} |\varphi_1^T \hat{\vartheta}_1|}, \quad (9)$$

then

$$\sup_{t \geq \bar{T}} |z_1(t)| \leq \rho.$$

- (3) If $\gamma_i := \gamma / \sqrt{r - i + 1/2}$, $i = 1, \dots, r$, for some disturbance attenuation level $\gamma > 0$, then the following disturbance attenuation inequality is satisfied: $\forall t \geq 0$

$$\int_0^t \sum_{i=1}^r \left[\beta_i z_i^2 + \gamma^2 \eta \frac{|\Phi_i^T \tilde{\xi}_i|^2}{|h_i|^2} \right] ds \leq \gamma^2 \int_0^t \sum_{i=1}^r \left| w_i + \frac{h_i(e_i^g + x_{i+1} e_i^b)}{|h_i|^2} \right|^2 ds + \sum_{i=1}^r (z_i^2(0)/2 + \gamma^2 |\tilde{\xi}_i(0)|^2),$$

where $\tilde{\xi}_i := [\tilde{\vartheta}_i^T, \tilde{\theta}_i^T]^T$. Moreover, if $e_i^g = e_i^b = 0$, and $w_i \in \mathcal{L}_2$, $i = 1, \dots, r$,

then the tracking error converges to zero, i.e.,

$$\lim_{t \rightarrow \infty} z_1(t) = 0.$$

(4) If, for any $i \in \{1, \dots, r\}$, $e_i^g = e_i^b = 0$, $w_i \in \mathcal{L}_2$, and

$$\lim_{t \rightarrow \infty} \lambda_{\min}(\Sigma_i^{-1}(t)) = \infty, \quad (10)$$

then $\lim_{t \rightarrow \infty} |\tilde{\xi}_i(t)| = 0$.

PROOF. The first part readily follows from the Remark 7 and Assumption 4. To prove the second part of the theorem, we note that $\|z_i(t)\| \leq M_i$, $\|P_i(\hat{\xi}_i(t))\| \leq 1$, $i = 1, \dots, r$, from the first part of the theorem. Now, from (8) and (9), whenever $|z_1(t)| \geq \rho$, clearly $\dot{V}_1 < 0$. This indicates that if $|z_1(\tilde{t})| \leq \rho$ for any $\tilde{t} \geq 0$, then $|z_1(t)| \leq \rho$ for all $t \geq \tilde{t}$. Now, if $\rho < |z_1(0)| \leq M_1$, suppose that to the contrary $|z_1(t)| > \rho$ for all $0 \leq t \leq \bar{T}$. Then, integrating \dot{V}_1 from 0 to \bar{T} leads to the following contradiction: $|z_1(\bar{T})|^2/2 < \rho^2/2$. Hence, $|z_1(t)| \leq \rho$ for all $t \geq \bar{T}$, proving the second part. To prove the third part, we note that the time derivative of $W_i := \tilde{\xi}_i^T \Sigma_i^{-1} \tilde{\xi}_i$ is upper-bounded by:

$$\frac{d}{dt} (W_i(t)) \leq -\eta \frac{|\Phi_i^T \tilde{\xi}_i|^2}{|h_i|^2} - |v_i|^2 + \left| w_i + \frac{h_i(e_i^g + x_{i+1}e_i^b)}{|h_i|^2} \right|^2 \quad (11)$$

It now readily follows from (7) and (11) that the time-derivative of $U := \sum_{i=1}^r (V_i + \gamma^2 W_i)$ is upper-bounded by:

$$\begin{aligned} \dot{U} \leq & -\sum_{i=1}^r \beta_i z_i^2 - \gamma^2 \sum_{i=1}^r \eta |\Phi_i^T \tilde{\xi}_i|^2 / |h_i|^2 + \sum_{i=1}^r ((r + 1/2 - i)\gamma_i^2 - \gamma^2) |v_i|^2 \\ & + \gamma^2 \sum_{i=1}^r \left| w_i + h_i(e_i^g + x_{i+1}e_i^b) / |h_i|^2 \right|^2. \end{aligned} \quad (12)$$

With the parameter choices made as in the third part, the integration of (12) yields the disturbance attenuation inequality. Moreover, if $e_i^g = e_i^b = 0$, and $w_i \in \mathcal{L}_2$, $i = 1, \dots, r$, then $z_1(t) \in \mathcal{L}_2$. This, with the uniform boundedness of $\dot{z}_1(t)$, implies that $z_1(t)$ converges to zero, which proves the third part. The last part is proven by integrating (11) to obtain

$$\lambda_{\min}(\Sigma_i^{-1}(t)) |\tilde{\xi}_i(t)|^2 \leq |\tilde{\xi}_i(0)|^2 + \int_0^t \left| w_i + h_i(e_i^g + x_{i+1}e_i^b) / |h_i|^2 \right|^2 ds, \quad \forall t \geq 0.$$

Since, under the hypothesis of the last part, the right hand side of the above inequality is finite and $\lim_{t \rightarrow \infty} \lambda_{\min}(\Sigma_i^{-1}(t)) = \infty$, then $|\tilde{\xi}_i(t)|$ must converge to zero. \square

6 Full State Measurements

We now consider the state-feedback case, where the derivatives of the state variables are not available to the controller. In this case, Identifier 1 cannot be used, but Identifier 2 can. Here, we use exactly the same controller designed in the previous section, but with the parameter estimates generated by Identifier 2, and obtain the counterparts of the results stated in Theorem 8. First, \tilde{x}_i is defined as $\tilde{x}_i := (x_i - \hat{x}_i)/\epsilon$, $i = 1, \dots, r$, and the dynamics of Identifier 2 are rewritten as:

$$\begin{aligned} \dot{\hat{\xi}}_i &= \mathcal{P}_i \Sigma_i \Phi_i \tilde{x}_i / |h_i|, \quad \mathcal{P}_i := \mathcal{P}[\tau_i, P_i(\hat{\xi}_i), \Sigma_i] \\ \dot{\tilde{x}}_i &= (h_i^T v_i - |h_i| \tilde{x}_i) / \epsilon, \quad \tilde{x}_i(0) = 0 \\ v_i &:= w_i + h_i((\varphi_i^T \hat{\theta}_i + e_i^b) x_{i+1} + \phi_i^T \tilde{\theta}_i + e_i^g) / |h_i|^2, \end{aligned} \quad (13)$$

where $\epsilon > 0$ is a small parameter. With this, the dynamics of the transformed variables z_i , $i = 1, \dots, r$, can be computed as:

$$\begin{aligned} \dot{z}_i &= \varphi_i^T \hat{\vartheta}_i z_{i+1} - \beta_i z_i - \varphi_{i-1}^T \hat{\vartheta}_{i-1} z_{i-1} + h_i^T v_i + \sum_{j=1}^{i-1} (p_{ij}^T v_j + s_{ij} \tilde{x}_j) \\ &\quad - |h_i|^2 z_i / (2\gamma_i^2) - \sum_{j=1}^{i-1} (|p_{ij}|^2 + \bar{q}_{ij}^2) z_i / (2\gamma_j^2) \\ \varphi_0 &= \hat{\vartheta}_0 = z_0 = z_{r+1} = 0 \end{aligned}$$

where z_i , β_i , p_{ij} , \bar{q}_{ij} are defined as in the previous section, and

$$s_{ij} := -\frac{\partial \alpha_{i-1}}{\partial \hat{\xi}_j^T} \frac{\mathcal{P}_j \Sigma_j \Phi_j}{|h_j|} - \sum_{m=j+1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial z_m} s_{mj}$$

such that $s_{ij}^2 \leq \bar{q}_{ij}^2$. Finally, let $V_i(t)$ be defined as in the previous section, i.e., $V_i(t) = z_i^2(t)/2$, $i = 1, \dots, r$. The time-derivative of $V_i(t)$ satisfies the bound:

$$\begin{aligned} \dot{V}_i &\leq \varphi_i^T \hat{\vartheta}_i z_i z_{i+1} - \beta_i z_i^2 + \sum_{j=1}^i \frac{\gamma_j^2 |v_j|^2}{2} + \sum_{j=1}^{i-1} \frac{\gamma_j^2 \tilde{x}_j^2}{2} - \varphi_{i-1}^T \hat{\vartheta}_{i-1} z_{i-1} z_i, \quad \text{or (14)} \\ \dot{V}_i &\leq \varphi_i^T \hat{\vartheta}_i z_i z_{i+1} - \frac{b_i}{\varphi_i^T \hat{\vartheta}_i} \beta_i z_i^2 + \gamma_i^2 |\bar{v}_i|^2 / 2 + \sum_{j=1}^{i-1} \gamma_j^2 [|v_j|^2 + \tilde{x}_j^2] / 2 \\ &\quad - \varphi_{i-1}^T \hat{\vartheta}_{i-1} z_{i-1} z_i \end{aligned} \quad (15)$$

where $\bar{v}_i := w_i + h_i((\varphi_i^T \tilde{\vartheta}_i + \check{e}_i)(z_{i+1} + \bar{\alpha}_i) + \phi_i^T \tilde{\theta}_i + e_i) / |h_i|^2$.

This now leads to the following counterpart of Theorem 8:

Theorem 9 Consider the closed-loop system described by (1), (4), and the designed controller, under Assumptions 1,...,6. Assume that $|z_i(0)| \leq M_i$, $|x_z(0)| \leq M_{z0}$, and the design functions β_i satisfy the lower bounds specified in Remark 7. Then:

- (1) $\|z_i(t)\| \leq M_i$, $\|P_i(\hat{\xi}_i(t))\| \leq 1$, $\|\tilde{x}_i(t)\| \leq M_{v_i}$, $i = 1, \dots, r$, $\|x_z(t)\| \leq M_z$.
(2) Given $\bar{T} > 0$, and $\rho > 0$ such that $\rho \leq M_1$, if

$$\beta_1 > \frac{M_1^2 - \rho^2 + 2 \max_{\bar{B}_1} (|\varphi_1^T \hat{\vartheta}_1| M_1 M_2 + \gamma_1^2 |\bar{v}_1|^2 / 2) \bar{T}}{2\rho^2 \bar{T} k_b / \max_{\bar{B}_1} |\varphi_1^T \hat{\vartheta}_1|}, \quad (16)$$

then

$$\sup_{t \geq \bar{T}} |z_1(t)| \leq \rho.$$

- (3) If $\gamma_i := \gamma / \sqrt{2(r-i+1)}$, $i = 1, \dots, r$, for some $\gamma > 0$, then the following disturbance attenuation inequality is satisfied: $\forall t \geq 0$

$$\begin{aligned} & \int_0^t \sum_{i=1}^r \left[\beta_i z_i^2 + \gamma^2 \eta \frac{|\Phi_i^T \tilde{\xi}_i|^2}{|h_i|^2} + \frac{\gamma_i^2 \tilde{x}_i^2}{2} \right] ds \\ & \leq \gamma^2 \int_0^t \sum_{i=1}^r \left| w_i + \frac{h_i(e_i^g + x_{i+1} e_i^b)}{|h_i|^2} \right|^2 ds + \sum_{i=1}^r (z_i^2(0)/2 + \gamma^2 |\tilde{\xi}_i(0)|^2) \\ & \quad + O(\epsilon)t. \end{aligned}$$

- (4) If, for any $i \in \{1, \dots, r\}$, $e_i^g = e_i^b = 0$, $w_i \in \mathcal{L}_2$, and

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \lambda_{\min}(\Sigma_i^{-1}(t)) \geq \delta_i \quad (17)$$

for some persistency of excitation level $\delta_i > 0$, then

$$\limsup_{t \rightarrow \infty} |\tilde{\xi}_i(t)|^2 \leq O(\epsilon).$$

PROOF. To prove the first part, let $V(t) := \max\{z_1^2(t)/M_1^2, \dots, z_r^2(t)/M_r^2\}$. Clearly, $V(0) \leq 1$. Let $t^* > 0$ be the first time, if ever, such that $V(t^*) = 1$. Obviously, $|z_i(t)| \leq M_i$, $P_i(\hat{\xi}_i) \leq 1$, $i = 1, \dots, r$, and $|x_z(t)| \leq M_z$, $\forall t \in [0, t^*]$. This implies $|v_i| \leq M_{v_i}$ when $t \in [0, t^*]$, which, from (13), requires $|\tilde{x}_i(t)| \leq M_{v_i}$ when $t \in [0, t^*]$. Now, because the design functions β_i satisfy the lower bounds specified in Remark 7, the upper right derivative of $V(t)$ at $t = t^*$ is negative, which proves the first part. The proof of the second part is completely analogous to the proof of the second part in Theorem 8, and hence

it is omitted. To prove the third part, let $W_i(t)$ be defined by

$$W_i := \tilde{\xi}_i^T \Sigma_i^{-1} \tilde{\xi}_i + \frac{\epsilon}{|h_i|} \left(\tilde{x}_i - \frac{\Phi_i^T \tilde{\xi}_i}{|h_i|} \right)^2.$$

The time derivative of $W_i(t)$ along the dynamics of Identifier 2 is upper-bounded by:

$$\dot{W}_i \leq -\eta \frac{|\Phi_i^T \tilde{\xi}_i|^2}{|h_i|^2} - \frac{|v_i|^2 + \tilde{x}_i^2}{4} + \left| w_i + \frac{h_i(e_i^g + x_{i+1}e_i^b)}{|h_i|^2} \right|^2 + O(\epsilon) \quad (18)$$

It now readily follows from (14) and (18) that the time-derivative of $U := \sum_{i=1}^r (V_i + \gamma^2 W_i)$ is upper-bounded by:

$$\begin{aligned} \dot{U} &\leq - \sum_{i=1}^r \beta_i z_i^2 - \gamma^2 \sum_{i=1}^r \eta |\Phi_i^T \tilde{\xi}_i|^2 / |h_i|^2 - \sum_{i=1}^r \gamma_i^2 \tilde{x}_i^2 / 2 \\ &\quad + \sum_{i=1}^r ((r+1-i)\gamma_i^2 / 2 - \gamma^2 / 4) [|v_i|^2 + \tilde{x}_i^2] \\ &\quad + \gamma^2 \sum_{i=1}^r \left| w_i + h_i(e_i^g + x_{i+1}e_i^b) / |h_i|^2 \right|^2 + O(\epsilon). \end{aligned} \quad (19)$$

With the parameter choices made as in the third part, the integration of (19) yields the disturbance attenuation inequality. The last part is proven by integrating (18), to obtain

$$\begin{aligned} &\frac{1}{t} \lambda_{\min}(\Sigma_i^{-1}(t)) \sup_{s \geq T} |\tilde{\xi}_i(s)|^2 \\ &\leq \frac{1}{t} |\tilde{\xi}_i(0)|^2 + \frac{1}{t} \int_0^t \left| w_i + h_i(e_i^g + x_{i+1}e_i^b) / |h_i|^2 \right|^2 ds + O(\epsilon), \quad \forall t \geq T, \end{aligned}$$

where $T > 0$ is an arbitrary number. Under the hypothesis of the last part, the above inequality yields $\sup_{s \geq T} |\tilde{\xi}_i(s)|^2 \leq O(\epsilon)$, as $t \uparrow \infty$, where $T > 0$ is arbitrary. This implies $\inf_{T \geq 0} \sup_{s \geq T} |\tilde{\xi}_i(s)|^2 \leq O(\epsilon)$, which completes the proof. \square

We note that the first two parts of Theorem 9, which state the boundedness of the closed-loop signals and the achievement of an arbitrarily small tracking error, are very similar to those of Theorem 8. However, the disturbance attenuation and parameter convergence results, which are stated in the third and fourth parts of Theorem 9, are corrupted by $O(\epsilon)$ terms, unlike the corresponding results stated in Theorem 8. Hence, the disturbance attenuation and parameter convergence results obtained with full state and derivative

measurements can be regarded as the limiting cases of those with full state measurements as $\epsilon \downarrow 0$. Finally, we recall that the parameter convergence results stated in Theorems 8 and 9 require the satisfaction of some persistency of excitation conditions (10) and (17). Intuitively, these conditions can be satisfied if the external disturbances, which are unknown, are sufficiently rich. However, there is a more direct way of satisfying (10) and (17) by making the reference signal sufficiently exciting, which is the subject of the next section.

7 Persistency of Excitation

We now address the issue of persistency of excitation, and present a scheme whereby the satisfaction of (10) or (17) is ensured, when the virtual control coefficients are known and the basis functions ϕ_i are Gaussian. The method presented below involves prescribing a persistent reference signal, and hence it does not apply to the set-point tracking case. It was shown earlier in [24] that if the state trajectory visits the centers of the basis functions regularly, then a relevant persistency of excitation condition is satisfied; see Lemmas 1 and 2 in [24]. What we present below is in the same spirit, and can be viewed as a modification of Lemma 2 of [24].

Lemma 10 *Consider the covariance dynamics of Identifiers 1 and 2, described by*

$$\dot{\Sigma} = (\eta - 1)\Sigma\Phi\Phi^T\Sigma/|h|^2, \quad \Sigma(0) = I, \quad \text{for some } 0 < \eta \leq 1,$$

and assume that the basis function Φ has the following structure:

$$\Phi := \left[\exp(-\sigma|x - c^1|^2), \dots, \exp(-\sigma|x - c^d|^2) \right]^T \quad (20)$$

for some positive number σ , some vectors c^j (called the centers of the basis functions), $j \in \{1, \dots, d\}$, with $c^j \neq c^k$ for all $j, k \in \{1, \dots, d\}$, $j \neq k$, and some positive integers d , where the subscript i is dropped to ease the notation. Further, assume that $\|x(t)\| \leq M_x$, for some $M_x > 0$, and $\sup_{t \geq T} |x(t) - r(t)| \leq \rho$, for some $T \geq 0$, $\rho > 0$, and vector valued signal $r(t)$ such that $r(t)$ visits the centers of Φ on a regular basis, and $\|\dot{r}(t)\| \leq M_{dr}$, for some $M_{dr} > 0$. More precisely, corresponding to every center c^j , there exist time instants $t_{c_j}^k \in [kt_p, (k+1)t_p]$, for some $t_p > 0$ and for all $k \in \{0, 1, 2, \dots\}$, such that $r(t_{c_j}^k) = c^j$. If ρ is sufficiently small, then

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \lambda_{\min}(\Sigma^{-1}(t)) \geq (1 - \eta)\delta / \sup_{|x| \leq M_x} |h(x)|^2, \quad \text{for some } \delta > 0.$$

for some persistency of excitation level $\delta > 0$, which is independent of the derivative of $x(t)$.

Proof of Lemma 10 is not included here because of space limitations, but it can be found in [2]. Lemma 10 states roughly that if the state trajectory visits the small neighborhoods of the centers of the radial basis functions, then some persistency of excitation level is guaranteed. Therefore, it may be possible to choose an appropriate reference signal $y_d(t)$ such that the conditions of Lemma 10 are satisfied. Although this does not seem to be an easy task for the general system (1), it could easily be achieved for the special case where the unknown nonlinear functions depend only on the first state x_1 ; see [3].

8 Output Measurements

In this section, we consider the counterpart of the robust tracking problem formulated in Section 3 for the case where only the scalar output variable is measured and used for control purposes. To obtain explicit results, the nonlinearities in the system model are taken to be dependent only on the output variable. Accordingly, we have the general dynamics (in strict output-feedback form):

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = A(x_1) \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0_{r-1 \times 1} \\ \varphi(x_1)\vartheta \\ 0_{n-r \times 1} \end{bmatrix} u + \begin{bmatrix} f_1(x_1) + g_1(x_1) + h_1^T(x_1)w_1 \\ \vdots \\ f_n(x_1) + g_n(x_1) + h_n^T(x_1)w_n \end{bmatrix}$$

$y = x_1$ (21)

The nonlinear function $A(x_1)$ is known, and has the following structure:

$$A(x_1) = \begin{bmatrix} 0_{n \times 1} & \left[\begin{array}{cccc|c} b_1(x_1) & 0 & \dots & 0 & \\ a_{22}(x_1) & b_2(x_1) & \dots & 0 & \\ \vdots & \ddots & \ddots & \vdots & \\ a_{r-1,2}(x_1) & \dots & a_{r-1,r-1}(x_1) & b_{r-1}(x_1) & \\ \hline a_{r,2}(x_1) & \dots & \dots & \dots & a_{r,n}(x_1) \\ \hline & & & A^{(2)}(x_1) & \end{array} \right] \end{bmatrix}$$

The nonlinear functions $h_i(x_1)$, $f_i(x_1)$, $i = 1, \dots, n$, $\varphi(x_1)$ are also known, whereas the nonlinear functions $g_i(x_1)$, $i = 1, \dots, n$, and the parameter $\vartheta \in \mathcal{R}$

are unknown. The objective is to force the output y track a prespecified signal $y_d(t)$ by picking the control input u appropriately as a function y . The unknown nonlinear terms $g_i(x_1)$ are approximated in terms of a set of sufficiently smooth basis functions, $\phi_i(x_1)$, and the optimal parameters θ_i are defined very similarly to the ones defined in Section 3.

Since the extended parameter vector $\xi := [\vartheta, \theta_1^T, \dots, \theta_n^T]^T$ and the state vector $x_f = [x_2, \dots, x_n]^T$ are unknown, they have to be identified by using the online information. For this purpose, we first append the natural dynamics of ξ to (21), and rewrite the overall system as:

$$\begin{aligned} \begin{bmatrix} \dot{\xi} \\ \dot{x}_f \end{bmatrix} &= \begin{bmatrix} 0_{p \times 1} \\ F(x_1) \end{bmatrix} + \begin{bmatrix} 0_{p \times p} & 0_{p \times n-1} \\ \Phi(x_1, u) & A_o(x_1) \end{bmatrix} \begin{bmatrix} \xi \\ x_f \end{bmatrix} + \begin{bmatrix} 0_{p \times q} \\ H(x_1) \end{bmatrix} w_e \\ \dot{x}_1 &= f_1(x_1) + C^T(x_1)x_f + \phi_{1e}(x_1)^T \xi + h_{1e}(x_1)^T w_e \end{aligned} \quad (22)$$

where $w_e := w + [h_1 e_1^g / |h_1|^2, \dots, h_n e_n^g / |h_n|^2]$. Estimates for the unknown terms ξ and x_f , which can be viewed as the states of the time-varying system (22), can be generated by a state estimator provided by the standard H^∞ theory; see [26, chapter 7]. Incorporating into the estimator design the additional information that the unknown parameter ξ satisfies $P(\xi) \leq 0$, where P is a convex and radially unbounded function, we introduce the following modified versions of the estimators introduced in [5]:

Estimator 1:

$$\begin{aligned} \dot{\hat{\xi}} &= \text{Proj}[\tau, P(\hat{\xi}), \Sigma], \quad \tau := \Sigma(\phi_{1e} + \Psi^T C)v / |h_1| \\ \dot{\hat{x}}_f &= F + A_o \hat{x}_f + \Phi \hat{\xi} + \Psi \text{Proj}[\tau, P(\hat{\xi}), \Sigma] + \Pi C v / |h_1| \\ v &= (\dot{x}_1 - f_1 - C^T \hat{x}_f - \phi_{1e}^T \hat{\xi}) / |h_1| \end{aligned} \quad (23)$$

Estimator 2 (parameterized by some $\epsilon > 0$):

$$\begin{aligned} \dot{\hat{\xi}} &= \text{Proj}[\tau, P(\hat{\xi}), \Sigma], \quad \tau := \Sigma(\phi_{1e} + \Psi^T C)(x_1 - \hat{x}_1) / (\epsilon |h_1|) \\ \dot{\hat{x}}_1 &= f_1 + C^T \hat{x}_f + \phi_{1e}^T \hat{\xi} + |h_1|(x_1 - \hat{x}_1) / \epsilon, \quad \hat{x}_1(0) = x_1(0) \\ \dot{\hat{x}}_f &= F + A_o \hat{x}_f + \Phi \hat{\xi} + \Psi \text{Proj}[\tau, P(\hat{\xi}), \Sigma] + \Pi C(x_1 - \hat{x}_1) / (\epsilon |h_1|) \end{aligned} \quad (24)$$

where $\hat{x}_f(0)$ is arbitrary, $\hat{\xi}(0)$ is such that $P(\hat{\xi}(0)) \leq 0$, and

$$\begin{aligned}
\dot{\Sigma} &= (\eta - 1)\Sigma Q \Sigma, \quad Q = (\phi_{1e} + \Psi^T C)(\phi_{1e} + \Psi^T C)^T / |h_1|^2, \quad \Sigma(0) = I \\
\dot{\Psi} &= (A_o - \Pi C C^T / |h_1|^2)\Psi + \Phi - \Pi C \phi_{1e}^T / |h_1|^2, \quad \Psi(0) = 0 \\
\dot{\Pi} &= A_o \Pi + \Pi A_o^T - \Pi C C^T \Pi / |h_1|^2 + H H^T + \Pi, \quad \Pi(0) = \Pi_0
\end{aligned}$$

for some $0 < \eta \leq 1$ and $\Pi_0 > 0$.

We note that Estimator 1 requires the knowledge of the output as well as its derivative, whereas Estimator 2 requires the knowledge of the output only. By applying the backstepping design technique on the state estimates $\hat{x}_f(t)$ generated by (23) or (24), depending on the available information, we can construct output-feedback adaptive controllers that provide the same performance as the state-feedback adaptive controllers designed in Sections 5 and 6. More precisely, we can obtain the counterparts of Theorems 8 and 9 for the output-feedback case. These results are not included here because of space limitations; they can be found in [2], which also contains details of the output-feedback adaptive controller design.

Finally, it is possible to extend the results of Sections 5 and 6 to the partial state-feedback case. In this case, the signals x_1, \dots, x_M , where M is an integer satisfying $1 < M < n$, are assumed to be available for feedback, and the nonlinear functions in the system dynamics can be allowed to depend on x_1, \dots, x_M . However, since this last extension follows from the derivation of the scalar output-feedback case in a rather straight-forward way (at least conceptually), we do not discuss it in further detail in this paper.

9 Simulation Results

Here we present a simulation study to illustrate and numerically validate the results of Sections 5 and 6. The system considered is a third-order one of the form (1), where

$$\begin{aligned}
n &= 3, \quad r = 2 \\
b_1(x_1) &= 2(1 + 0.1x_1^2) \\
b_2(x_{[2]}, x_z) &= 2(1.1 + 0.1 \sin(x_2)) + 0.01(x_1^2 + x_2^2 + x_z^2) \\
g_1(x_1) &= x_1 + 0.01x_1^4 + 0.1 \sin(x_1) \\
g_2(x_{[2]}, x_z) &= x_1 + x_2 + 0.1x_1^2 + 0.1x_2^2 + 0.1x_z \\
g_z(x_{[2]}, x_z) &= -10x_z + 0.5x_1^2 + 0.2 \cos x_1 \cos x_2 + h_z^T w_z \\
\varphi_1(x_1) &= \exp(-x_1^2/20), \quad \varphi_2(x_2) = \exp(-x_2^2/20) \\
\phi_1(x_1) &= [\exp(-(x_1 - 1)^2) \quad \exp(-(x_1 + 1)^2)]^T
\end{aligned}$$

$$\begin{aligned}
\phi_2(x_{[2]}) &= [\exp(-(x_1 - 1)^2 - (x_2 - 1)^2) \\
&\quad \exp(-(x_1 + 1)^2 - (x_2 + 1)^2)]^T \\
h_1(x_1) &= [1 \ x_1]^T, \quad h_2(x_z) = [0.1x_z \ 1]^T, \quad h_z(x_{[2]}) = [1 + x_1 \ x_2]^T \\
w_1(t) = w_2(t) = w_z(t) &= [0.5 \sin(5t) \ 0.1 \cos(20t) + 0.2 \sin(10t)]^T.
\end{aligned}$$

The unknown nonlinear functions $b_i, g_i, i \in \{1, 2\}$ are identified within $B_1 := \{x_1 \in \mathcal{R} : |x_1| \leq 1\}$, and $B_2 := \{x_{[2]}, x_z \in \mathcal{R}^3 : |x_1| \leq 1, |x_2| \leq 1.5, |x_z| \leq 2\}$, by using the basis functions φ_i , and ϕ_i , respectively. The optimal parameter values are numerically computed as: $\vartheta_1 = 2.2, \vartheta_2 = 2.4, \theta_1 = [1.0 \ -1.0], \theta_2 = [2.9 \ -1.5]$. We assume that these unknown parameters satisfy: $|\theta_i|^2 \leq 50$ and $\vartheta_i \in [1, 5], i \in \{1, 2\}$. Note that $\varphi_i(x^i)^T \bar{\vartheta}_i > k$, for some constant $k > 0$ and for all $\bar{\vartheta}_i \in [1, 5], x^i \in B_i, i \in \{1, 2\}$. From this, it is straight-forward to show that the unknown parameters satisfy Assumption 3 with the following functions:

$$\begin{aligned}
P_1(\xi_1) &:= ((\vartheta_1 - 3)^2/4 + |\theta_1|^2/50 - 1)/0.2 \leq 0 \\
P_2(\xi_2) &:= ((\vartheta_2 - 3)^2/4 + |\theta_2|^2/50 - 1)/0.2 \leq 0.
\end{aligned}$$

The reference signal is $y_d(t) = \sin(t)$. The designed controller with Identifier 2, which requires full state measurements only², is used with the following parameters: $\beta_1 = 5.0, \beta_2 = 1.0, \gamma_1^2 = \gamma_2^2 = 5.0, \eta = 1, \epsilon = 0.1$. The simulation results corresponding to the closed-loop system are shown in Figures 1 and 2. As it should be clear from the figures, the controller keeps all the signals bounded. Figure 1 shows that, after an initial deviation, the output of the system always stays within a small neighborhood of the reference signal. The parameter estimates approach their optimal values reasonably fast, and then experience small oscillations, as shown in Figure 2. This indicates that an adequate identification of the uncertain system is achieved. Overall, these numerical results are in agreement with the theoretical results presented in Sections 5 and 6, and illustrate the effectiveness of the proposed controller.

10 Conclusions

A neural-net based adaptive controller design has been presented for uncertain strict-feedback systems with unknown virtual control coefficients. The controller design procedure involves neural-network based approximation of nonlinearities, the use of some H^∞ based identifiers, the backstepping design methodology, and the use of radially unbounded functions. The resulting controller achieves the output tracking of a reference signal arbitrarily well at

² The simulation results with Identifier 1 turn out to be very similar to those with Identifier 2, and hence are not included here.

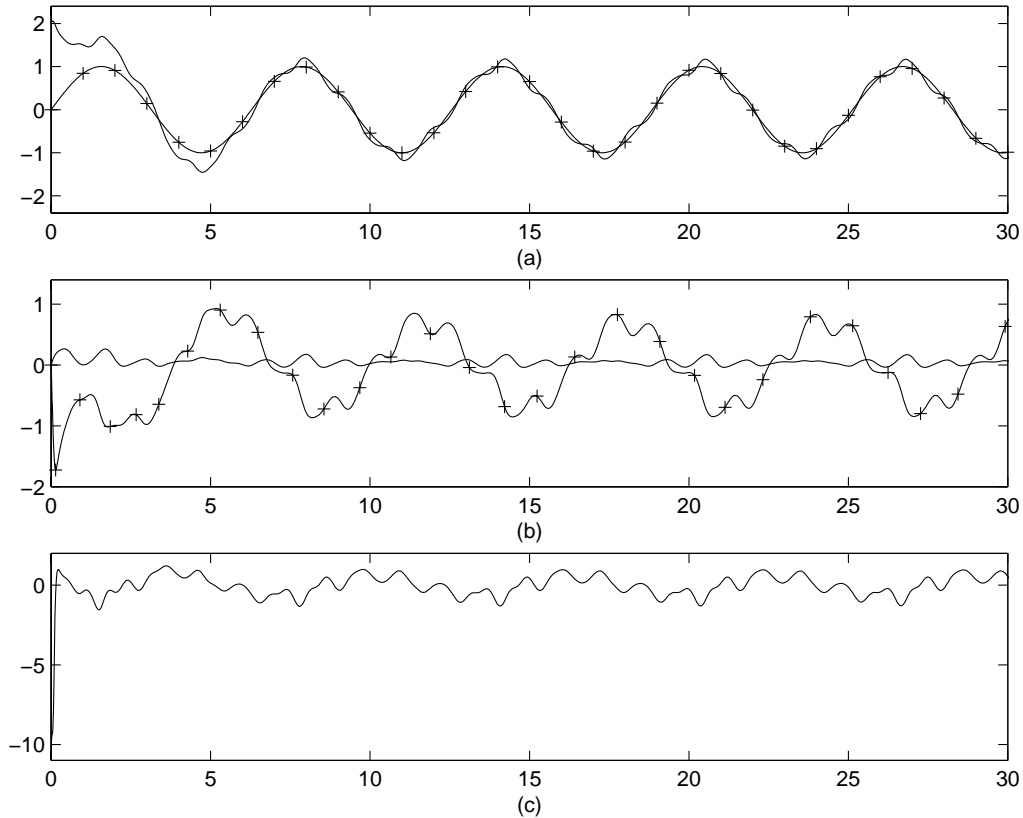


Fig. 1. Performance of the designed controller with full state measurements.
(a) y_d (+), x_1 (-); (b) x_2 (+), x_3 (-); (c) u .

the expense of increased control effort, as stated in Theorems 8 and 9. Also, the closed-loop signals remain bounded and satisfy a relevant disturbance attenuation inequality. These results are also extended to the case where only the output of the plant is available for feedback. In this case, the system model is inescapably less general, because for tractability the nonlinearities can be allowed to depend only on the observed variables, which in this case is the scalar output variable. Under this structural restriction, we estimate the unmeasured states as well as the unknown parameters that characterize an appropriate approximation of the nonlinearities, and subsequently apply the backstepping design technique on the estimates of the unmeasured states to design adaptive disturbance attenuating controllers with output feedback; for details see [2].

One extension of this work would be to obtain the counterparts of the results presented here for a strict (output) feedback system perturbed by random external disturbances. For such a system, which is commonly modeled by a set of Itô stochastic differential equations, the controllers presented in this paper may not yield a satisfactory performance because of the quadratic variation terms resulting from the Itô differentiation rule. Hence, a different controller design strategy than the one presented in this paper needs to be developed to

cope with the quadratic variation terms in that case. Another extension is to design a decentralized robust control scheme for a set of strict feedback systems, which is motivated by some applications such as power system control. Both of these research topics mentioned above are currently under study.

Acknowledgements

This work was supported in part by the U.S. Department of Energy under Grant DOE-DEFG-02-94ER13939.

References

- [1] D.Seto, A.M.Annaswamy, and J.Baillieul. Adaptive control of nonlinear systems with a triangular structure. *IEEE Transactions on Automatic Control*, 39(7):1411–1428, July 1994.

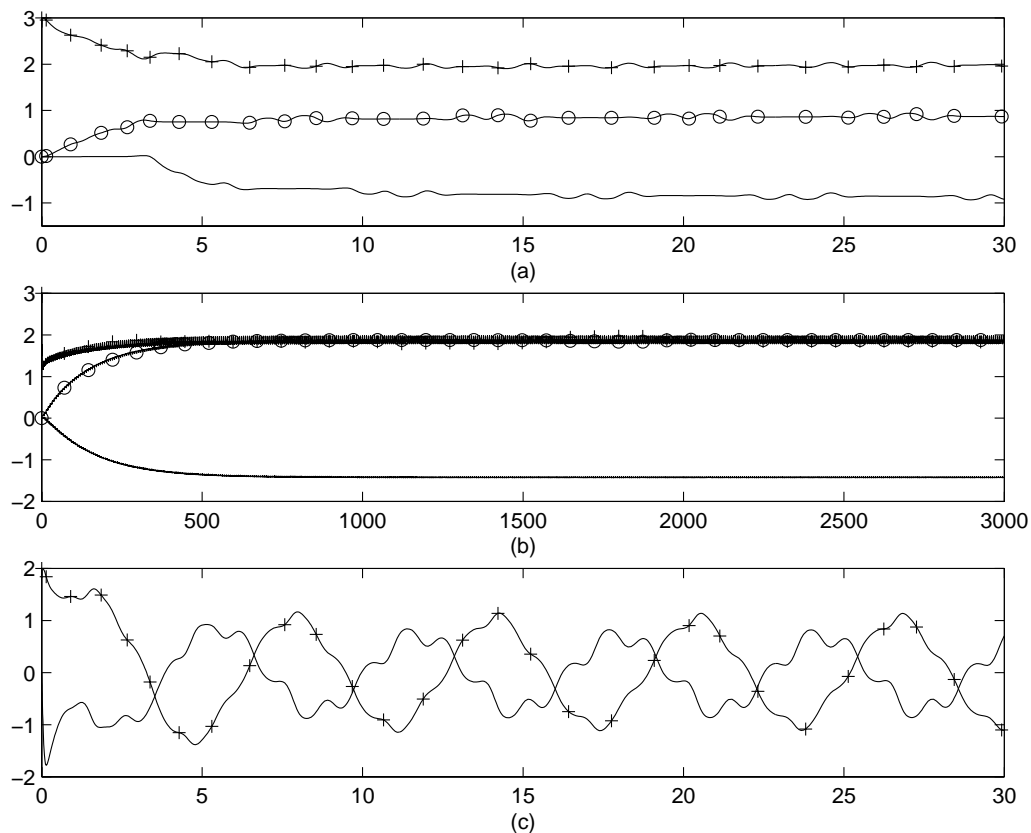


Fig. 2. Parameter estimates generated by Identifier 2.

(a) \hat{v}_1 (+), $\hat{\theta}_{11}$ (o), $\hat{\theta}_{12}$ (-); (b) \hat{v}_2 (+), $\hat{\theta}_{21}$ (o), $\hat{\theta}_{22}$ (-); (c) \hat{x}_1 (+), \hat{x}_2 (-).

- [2] G.Arslan. *Robust Adaptive Control of Deterministic and Stochastic Strict-Feedback-Systems*. Ph.D. thesis, Dept. of Electrical and Computer Engineering, and Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, 2001.
- [3] G.Arslan and T.Başar. Robust identification for output tracking in systems with unknown nonlinearities. In *Proc. of the 1999 ACC*, pages 1057–1061, San Diego, CA, June 1999.
- [4] G.Didinsky, Z.Pan, and T.Başar. Parameter identification for uncertain plants using H^∞ methods. *Automatica*, 39(9):1227–1250, September 1995.
- [5] I.E.Tezcan and T.Başar. Disturbance attenuating adaptive controllers for parametric strict feedback nonlinear systems with output measurements. *ASME Journal of Dynamic Systems, Measurement and Control*, 121:48–57, Mar 1999.
- [6] J.Park and I.W.Sandberg. Approximation and radial-basis-function networks. *Neural Computation*, pages 305–316, 1993.
- [7] J.W.Helton and M.R.James. Extending H_∞ control to nonlinear systems. *SIAM Frontiers in Applied Mathematics*, 1999.
- [8] K.Ezal, A.R.Teel, P.Kokotović, and T.Başar. Disturbance attenuating output-feedback control of nonlinear systems with local optimality. In *Proc. of the 1999 ACC*, pages 2578–2583, San Diego, CA, June 1999; also to appear in *Automatica*, June 2001.
- [9] K.Ezal, Z.Pan, and P.V.Kokotović. Locally optimal and robust backstepping design. *IEEE Transactions on Automatic Control*, 45(2):260–271, February 2000.
- [10] K.Hornik, M.Stinchcombe, and H.White. Universal approximation of an unknown mapping and its derivatives using multilayer feedforward networks. *Neural Networks*, 3:551–560, 1990.
- [11] K.S.Narendra and S.Mukhopadhyay. Adaptive control using neural networks and approximate models. *IEEE Transactions on Neural Networks*, 8:475–485, 1997.
- [12] J.-S. Lin and I.Kanellakopoulos. Modular adaptive design of active suspensions. In *Proc. of the 36th CDC*, pages 3626–3631, San Diego, CA, December 1997.
- [13] M.Krstić, I.Kanellakopoulos, and P.Kokotović. *Nonlinear and Adaptive Control Design*. Wiley, New York, 1995.
- [14] M.Krstić, J.M.Protz, J.D.Paduano, and P.V.Kokotović. Backstepping designs for jet engine stall and surge control. In *Proc. of the 34th CDC*, pages 3049–3055, New Orleans, LA, 1995.
- [15] M.M.Polycarpou. Stable adaptive neural control scheme for nonlinear systems. *IEEE Transactions on Automatic Control*, 41:447–451, 1996.

- [16] M.M.Polycarpou and M.J.Mears. Stable adaptive tracking of uncertain systems using nonlinearly parameterized on-line approximators. *International Journal of Control*, 70(3):363–84, June 1998.
- [17] M.M.Polycarpou and P.A.Ioannou. A robust adaptive nonlinear control design. *Automatica*, 32(3):423–427, 1996.
- [18] M.R.James and J.S.Baras. Robust H_∞ output feedback control for nonlinear systems. *IEEE Transactions on Automatic Control*, 40:1007–1017, 1995.
- [19] M.T.Alrifai, J.H.Chow, and D.A.Torrey. A backstepping nonlinear control approach to a switched-reluctance motors. In *Proc. of the 1998 CDC*, pages 4652–4657, 1998.
- [20] P.Cardaliaguet and G.Euvrard. Approximation of a function and its derivative with a neural network. *Neural Networks*, 5:207–220, 1992.
- [21] L. Praly, G. Bastin, J. P. Pomet, and Z. P. Jiang. Adaptive stabilization of nonlinear systems. In P. V. Kokotović, editor, *Foundations of Adaptive Control*, pages 347–433. Springer-Verlag, Berlin, 1991.
- [22] R.M.Sanner and J-J.E.Slotine. Gaussian networks for direct adaptive control. *IEEE Transactions on Neural Networks*, 3(6):837–863, November 1992.
- [23] S-H.Yu and A.M.Annaswamy. Stable neural controllers for nonlinear dynamic systems. *Automatica*, 34(5):641–650, sep 1995.
- [24] S.Lu and T.Başar. Robust nonlinear system identification using neural-network models. *IEEE Transactions on Neural Networks*, 9(3):407–429, May 1998.
- [25] E. D. Sontag and Y. Wang. New characterization of input-to-state stability. *IEEE Transactions on Automatic Control*, 41(9):1283–1294, September 1996.
- [26] T.Başar and P.Bernhard. *H^∞ – Optimal Control and Related Minimax Design Problems: A Dynamic Game Approach*. Birkhauser, Boston, 2nd ed., 1995.
- [27] T.I.Fossen and Å.Grøvlén. Nonlinear output feedback control of dynamically positioned ships using vectorial observer backstepping. *IEEE Transactions on Control Systems Technology*, 6(1):121–128, 1998.
- [28] T.Zhang, S.S.Ge, and C.C.Hang. Adaptive neural network control for strict-feedback nonlinear systems using backstepping design. In *Proc. of the 1999 ACC*, pages 1062–1066, San Diego, CA, June 1999.
- [29] Y.Guo, Z-P.Jiang, and J.Hill. Decentralized robust disturbance attenuation for a class of large-scale nonlinear systems. *System & Control Letters*, 37:71–85, 1999.
- [30] Y.Zhang and P-Y.Peng. Stable neural controller design for unknown nonlinear systems using backstepping. In *Proc. of the 1999 ACC*, pages 1067–1071, San Diego, CA, June 1999.
- [31] Z.Pan and T.Başar. Adaptive controller design for tracking and disturbance attenuation in parametric-strict-feedback nonlinear systems. *IEEE Trans. on Automatic Control*, 43:1066–1083, 1998.