# Extremal Distributions in Information Theory and Hypothesis Testing

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# Neyman Pearson Hypothesis Testing

Observations $\boldsymbol{X} = \{X_t : t = 1, 2, \ldots N\}$

$X_t$ takes values in finite alphabet $\mathcal{A}$

$X$ i.i.d. with marginal $\pi_j$ under $H_j$, $j = 0, 1$

Hypothesis test:

$$\phi(x) = 1 \text{ if } x \in A_1 \subset \mathcal{A}^N$$

Error Probabilities

$$\mathsf{P}_{\mathsf{e},0} = \mathsf{P}_0\{\phi(X) = 1\}, \quad \mathsf{P}_{\mathsf{e},1} = \mathsf{P}_1\{\phi(X) = 0\}$$

N-P Criterion: $\inf_{\phi} \mathsf{P}_{\mathsf{e},1}$ subject to $\mathsf{P}_{\mathsf{e},0} \leq \eta$

# Likelihood Ratio Test

Likelihood ratio

$$\ell(a) = \frac{\pi_{1,a}}{\pi_{0,a}}, \quad a \in \mathcal{A}$$

Optimum test (LRT)

$$A_1 = \left\{ x : \frac{1}{N} \sum_{t=1}^{N} \log \ell(X_t) \ \geq \ \lambda \right\}$$

Empirical distribution of $x$

$$\tilde{\mu}_a = \frac{1}{N} \sum_{t=1}^{N} \mathbb{I}_{\{X_t = a\}}$$

Alternative form for LRT:

$$A_1 = \{ x : \langle \tilde{\mu}, \log \ell \rangle \ \geq \ \lambda \}$$

# Robust Hypothesis Testing

$\pi_0$ and $\pi_1$ not known exactly

Uncertainty classes

$\pi_0 \in \mathbb{P}_0 \qquad \pi_1 \in \mathbb{P}_1$

Robust N-P Criterion

$$\inf_{\phi} \sup_{\pi_1 \in \mathbb{P}_1} \mathsf{P}_{e,1} \text{ subject to } \sup_{\pi_0 \in \mathbb{P}_0} \mathsf{P}_{e,0} < \eta$$

Robust test can be found for special uncertainty classes

$\epsilon$-contamination classes, band classes,
T-V neighborhoods, $p$-point classes, etc.

[Huber (65,73)]

Robust test is LRT between *least favorable distributions*

# Moment Classes

Uncertainty classes based on *fixing moments*

$$\mathbb{P}_j = \left\{ \pi : \langle \pi, f_i \rangle = c_i^j, i = 1, ..., n \right\}$$

Natural way to define uncertainty in practical scenarios (e.g. mean and variance constraints)

Markov (1890's)
Whitt (1976)
Smith (1995)
Bertsimas & Sethuraman (2000)
Vandenberghe, Boyd & Comanor (2004)

Applications in detection in sensor networks

*Robust hypothesis testing problem for moment classes does not fall under Huber framework*

# Asymptotic N-P Hypothesis Testing

Sequence of tests

$$\phi_N(x^N) = 1 \ \text{ if } x^N \in A_1^N$$

Number of observations $N \to \infty$

Error exponents:

$$\begin{aligned} \varepsilon_1 \ &:= \ -\liminf_{N\to\infty} \frac{1}{N} \log \mathsf{P}_1\{\phi_N(X^N) = 0\} \\ \varepsilon_0 \ &:= \ -\liminf_{N\to\infty} \frac{1}{N} \log \mathsf{P}_0\{\phi_N(X^N) = 1\} \end{aligned}$$

Asymptotic N-P problem:

$$\sup_{\phi} \varepsilon_1 \qquad \text{subject to} \qquad \varepsilon_0 \ge \eta$$

# Asymptotic N-P Hypothesis Testing

Optimal value of $\varepsilon_1$ :

$$\beta^* = \inf_{\mu \in \mathcal{Q}_\eta(\pi_0)} D(\mu \,\|\, \pi_1)$$

Hoeffding (1965)

Blahut (1974)

where

$$\mathcal{Q}_\eta(\pi_0) := \{\mu : D(\mu \,\|\, \pi_0) < \eta\}$$

Optimal test sequence

$$\begin{aligned} A_1^N &= \Big\{ x^N \in \mathcal{A}^N : \frac{1}{N}\sum_{t=1}^{N} \log \ell(x_t) \ \geq\ \eta - \beta^* \Big\} \\ &= \big\{ x^N \in \mathcal{A}^N : \langle \tilde{\mu}_N, \log \ell \rangle \ \geq\ \eta - \beta^* \big\} \\ &= \big\{ x^N \in \mathcal{A}^N : \langle \tilde{\mu}_N, \log \ell \rangle \ \geq\ \langle \mu^*, \log \ell \rangle \big\} \end{aligned}$$

# Asymptotic N-P Hypothesis Testing

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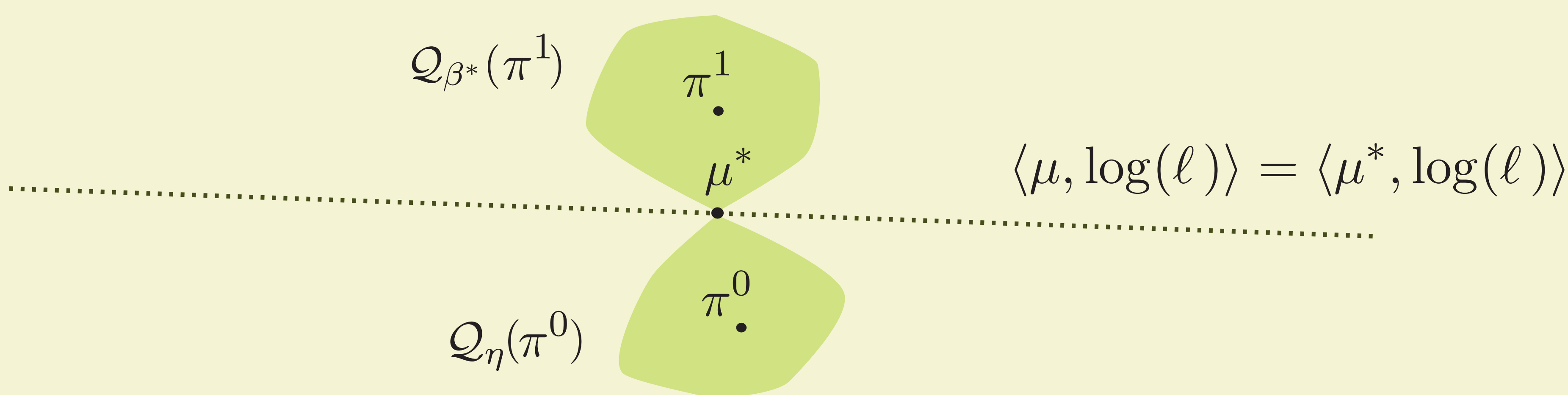


# Asymptotic Robust N-P Hypothesis Testing

## Moment Classes

$\pi_0$ and $\pi_1$ not known exactly: $\quad \pi_0 \in \mathbb{P}_0 \quad \pi_1 \in \mathbb{P}_1$

$$\mathbb{P}_j = \left\{ \pi : \langle \pi, f_i \rangle = c_i^j \,, \; i = 1, ..., n \right\}$$

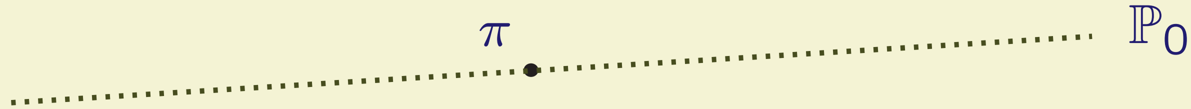
## Asymptotic Robust N-P Criterion

$$\sup_{\phi} \inf_{\pi_1 \in \mathbb{P}_1} \varepsilon_1 \;\; \text{subject to} \;\; \inf_{\pi_0 \in \mathbb{P}_0} \varepsilon_0 \; \geq \; \eta$$

# Divergence Sets

Divergence set containing $\mathbb{P}_0$ :

$$\mathcal{Q}_\eta(\mathbb{P}_0) = \bigcup_{\pi \in \mathbb{P}_0} \mathcal{Q}_\eta(\pi) = \bigcup_{\pi \in \mathbb{P}_0} \{\mu : D(\mu \,\|\, \pi) < \eta\}$$



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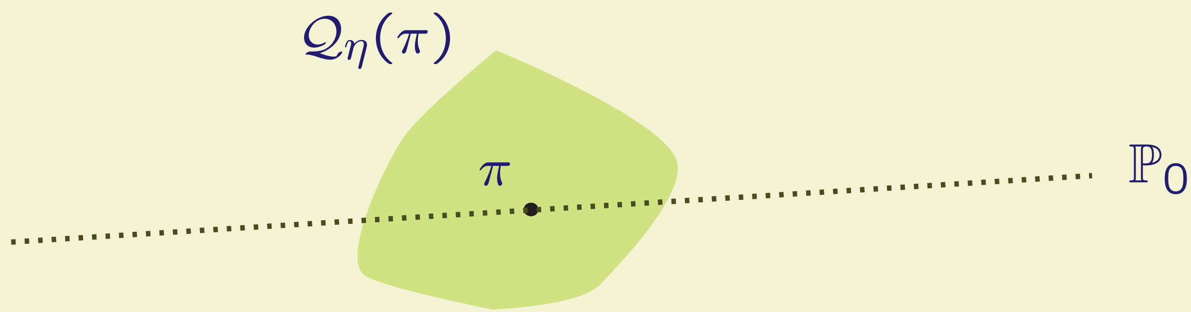

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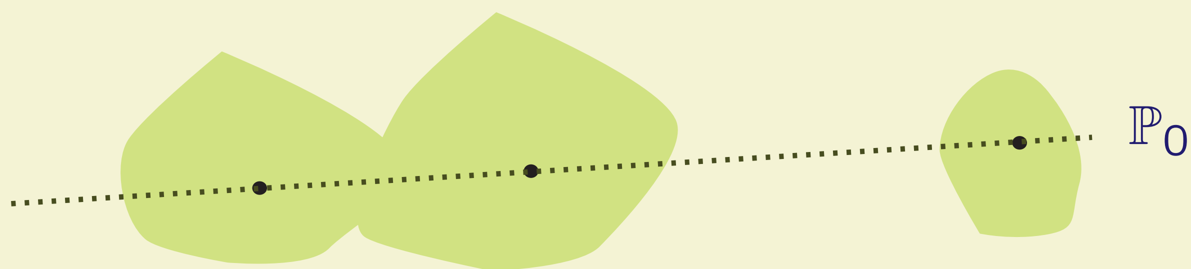

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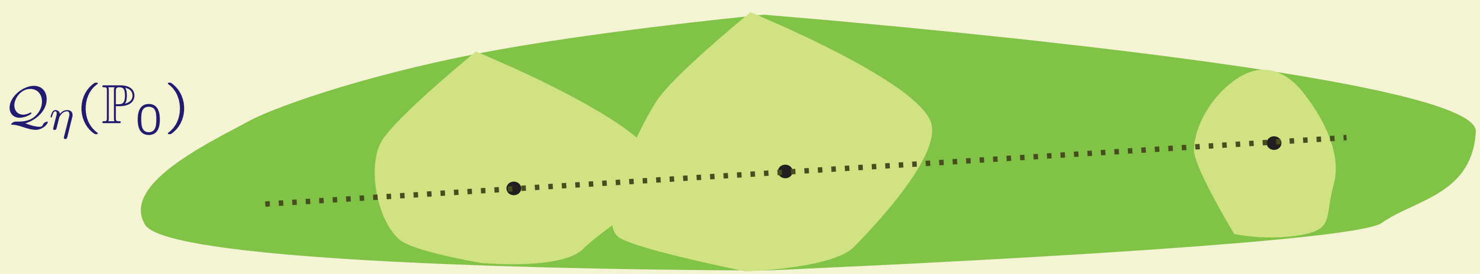

# Solution to the Robust Neyman-Pearson Problem

Optimal value of $\varepsilon_1$ : There exist $\pi_0^* \in \mathbb{P}_0, \pi_1^* \in \mathbb{P}_1$, and $\mu^*$ solving,

$$\beta^* = \inf_{\pi^1 \in \mathbb{P}_1} \inf_{\mu \in \mathcal{Q}_\eta(\mathbb{P}_0)} D(\mu \,\|\, \pi^1)$$

Optimal robust test: LRT between $\pi_1^*$ and $\pi_0^*$ :

$$\ell(a) = \frac{\pi_{1,a}^*}{\pi_{0,a}^*} = \frac{\gamma^\top f(a)}{\lambda^\top f(a)}$$

Optimal robust test sequence

$$\begin{aligned} A_1^N &= \left\{ x^N \in \mathcal{A}^N : \langle \tilde{\mu}_N, \log \ell \rangle \ \geq \ \eta - \beta^* \right\} \\ &= \left\{ x^N \in \mathcal{A}^N : \langle \tilde{\mu}_N, \log \ell \rangle \ \geq \ \langle \mu^*, \log \ell \rangle \right\} \end{aligned}$$

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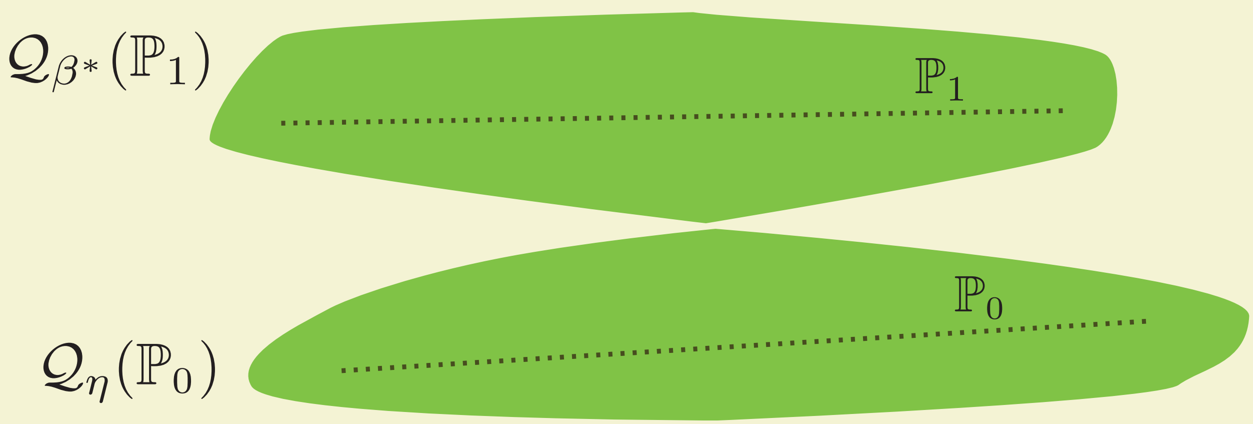

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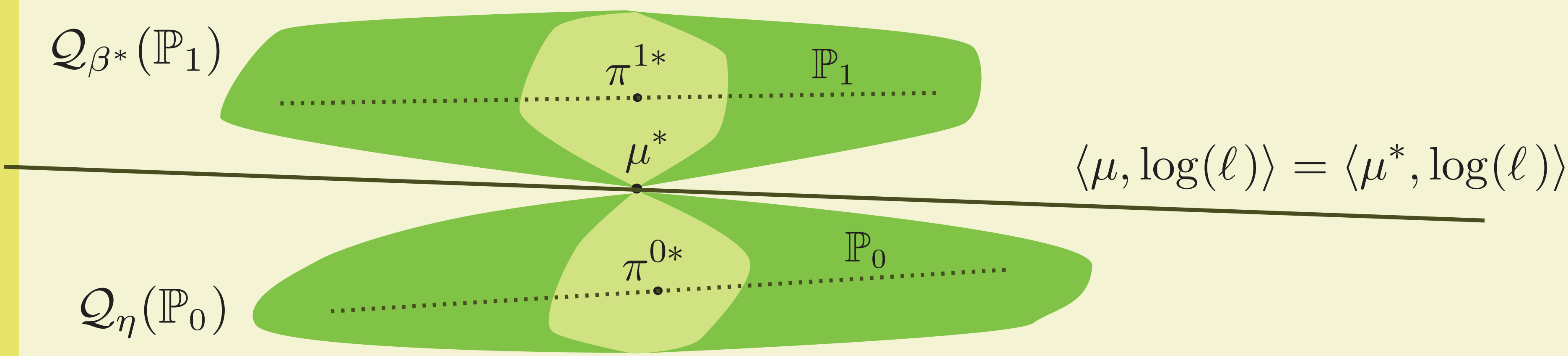

# Solution Structure

$$\ell(a) = \frac{\pi^*_{1,a}}{\pi^*_{0,a}} = \frac{\gamma^\top f(a)}{\lambda^\top f(a)}$$

Robust test is based on a log-linear combination of data

Optimization is convex: Optimal $\pi^*_0$, $\pi^*_1$, $\mu^*$ and $\lambda$ and $\gamma$ easily computed numerically

Worst case distributions typically discrete: Cutting plane algorithm effective

# Conclusions & Extensions

| | |
|---|---|
| Immediate extensions: | $M$-*ary* hypothesis testing |
| | Non-finite alphabets |
| | Asymmetric moment classes |
| Current research: | Non-i.i.d. observations |
| | Optimizing choice of constraint functions |
| | Applications |

# References


C. Pandit, S.P. Meyn, V.V. Veeravalli, *Robust Neyman-Pearson Hypothesis Testing Based on Moment Classes*, 2004

C. Pandit and S.P. Meyn, *Robust Measurement-Based Admission Control Using Markov's Theory of Canonical Distributions*, 2003

C. Pandit, and S.P. Meyn, *Extremal Distributions and Worst-Case Large-Deviation Bounds*, 2004.

J. Huang and S.P. Meyn, *Characterization and Computation of Optimal Distributions for Channel Coding*.
To appear, IEEE Trans. Info. Theory.


*Preprints available at http://black.csl.uiuc.edu/~meyn*