

# Optimal Estimation with Limited Measurements

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## Abstract

We consider a sequential estimation problem with two decision makers, or agents, who work as members of a team. One of the agents sits at an observation post, and makes sequential observations about the state of an underlying stochastic process for a fixed period of time. The observer agent upon observing the process makes a decision as to whether to disclose some information about the process to the other agent who acts as an estimator. The estimator agent sequentially estimates the state of the process. The agents have the common objective of minimizing a performance criterion with the constraint that the observer agent may only act a limited number of times.

## I. INTRODUCTION

Recursive estimation of a linear stochastic process with full and partial state information has been extensively studied in the literature [1]. In this paper, we introduce the problem of recursive estimation with *limited* information. More specifically, we consider estimating a stochastic process over a decision horizon of length  $N$  using only  $M \leq N$  measurements. Both the measurement and estimation of the process is carried out sequentially by two different decision makers, or agents, called the *observer* and the *estimator*,<sup>1</sup> respectively. Over the decision horizon of length  $N$ , the observer agent has exactly  $M$  opportunities to disclose some information about the process to the estimator. These information disclosures, or *transmissions*, are assumed to be error and noise free, and the problem is to jointly determine the best observation and estimation policies that minimize the average estimation error between the process and its estimate.

Estimation problems of this nature arise in many applications ranging from monitoring and control over wireless sensor networks [2], and scheduling of packet transmissions over time-allocation limited

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<sup>1</sup>As we show next, in a communication-theoretic setting we may call them an *encoder* and a *decoder*, respectively.

channels. For example, due to the power-limited nature of the wireless sensors, in most sensor net applications the wireless devices can only make a *limited* number of transmissions [3].

The rest of the paper is organized as follows. In Section II, we formally define the problem, and briefly discuss some of the potential applications. Section III discusses estimating an i.i.d. random sequence with a limited number of measurements. In Section IV, the results of Section III are extended to the Gauss-Markov case. We present some illustrative examples in Section V, and the paper ends with the concluding remarks of Section VI.

## II. PROBLEM STATEMENT

### A. Problem Definition

The problem of optimal estimation with limited measurements can be treated in the more general framework of a communication system with limited channel uses. For this purpose, consider the generic communication system whose block diagram is given in Figure 1 [4]. The source outputs some data  $b_k$  for  $0 \leq k \leq N-1$ , that needs to be communicated to the user over a channel. The data  $b_k$  are generated according to some *a priori* known stochastic process,  $\{b_k\}$ , which may be i.i.d., or correlated as in a Markov process. An encoder (or an observer) and a decoder (or an estimator) is placed after the source output and the channel output, respectively, to communicate the data to the user efficiently. In the most general case, the encoder/observer may have access to a noise-corrupted version of the source output:

$$z_k = b_k + v_k, \quad 0 \leq k \leq N-1$$

where  $\{v_k\}$  is an independent<sup>2</sup> noise process.

The main constraint is that the encoder/observer can access the channel only a *limited*,  $M \leq N$ , number of times. The goal is to design an observer-estimator pair,<sup>3</sup>  $(\mathcal{O}, \mathcal{E})$ , that will “causally” (or sequentially) observe/enconde the data measurements,  $z_k$ , and estimate/decode the channel output,  $y_k$ , so as to minimize the *average distortion* or *error* between the observed data,  $b_k$ , and estimated data,  $\hat{b}_k$ .

The channel is assumed to be memoryless, and is completely characterized by the conditional probability distribution  $P_c(y|x)$  on  $y \in \mathcal{Y}$  for each  $x \in \mathcal{X}$ , where  $\mathcal{X}$  and  $\mathcal{Y}$  are the set of allowable channel inputs, and the set of possible channel outputs, respectively.

<sup>2</sup>Independent across time and from the source output process  $b_k$ .

<sup>3</sup>Or depending on the application, an encoder-decoder pair  $(\mathcal{E}, \mathcal{D})$ .

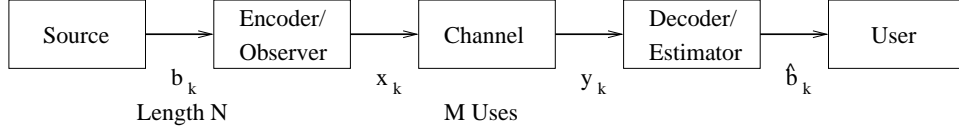


Fig. 1. Communication with limited channel use.

The average distortion  $D_{(M,N)}$  depends on the distortion measure and may vary depending on the underlying application. Some examples are the average mean-square error

$$D_{(M,N)} = E \left\{ \frac{1}{N} \sum_{k=0}^{N-1} (b_k - \hat{b}_k)^2 \right\} \quad (1)$$

or the Hamming (probability of error) distortion measure

$$D_{(M,N)} = E \left\{ \frac{1}{N} \sum_{k=0}^{N-1} \mathcal{I}_{b_k \neq \hat{b}_k} \right\} \quad (2)$$

where  $\mathcal{I}_S$  denotes the indicator function of the set  $S$ .

From a communication-theoretic standpoint, with the channel, source, and the distortion measure defined, we can formally state our main problem: Given a source and a memoryless channel, for a given decision-horizon  $N$ , and number of channel uses  $M$ , what is the minimum attainable value of the average distortion  $D_{(M,N)}$ ? This minimization is carried out over the choice of possible encoder-decoder (observer-estimator) pairs which are *causal*.

In this paper, we present a solution to this problem when the source process is i.i.d. with a continuous or discrete probability density function, and the encoder/observer has access to the noiseless or a noisy version of the source output. We assume that the channel is noiseless, and hence, it is completely characterized by the probability distribution  $P_c(y|x) = \delta(y-x)$ . We also present the solution to the case when the source process is Gauss-Markov.

## B. Example Applications

In most industrial wireless sensing applications, wireless sensors communicate over an RF (radio frequency) channel, and a significant amount of power is consumed to transmit the sensor measurements [3]. In most sensor net applications, the wireless sensors are battery-powered [3], and therefore, it is important to use the wireless channel only when it is necessary to extend the life of the wireless device as long as possible. The desired length of time the wireless device will be in operation can be related to the decision horizon  $N$  in some appropriate time unit, and the size of the battery installed in the sensor can be related to the possible number of transmissions or channel uses  $M$  (see Figure 2).

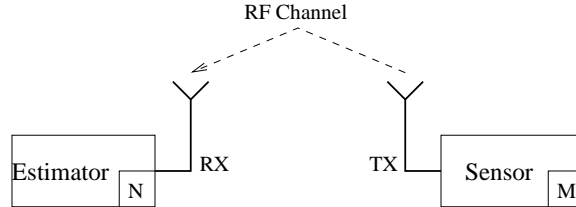


Fig. 2. Optimal transmission scheduling with limited channel access.

Hence, given an underlying performance criterion  $D_{(M,N)}$ , the problem is to design the best transmission schedule, and estimation policies for the wireless device and the remote monitoring station, respectively.

Another potential application is when a transmitting device (or source) needs to share the available transmission bandwidth with several other competing users, and therefore is time-allocation limited, such as in a time-division multiplexing (TDM) system. Say, for instance, the source has a message of size  $N$  packets, which is sequentially revealed to the encoder packet by packet, and due to the time-allocation limitation, the transmitter can only schedule  $M$  packets over the next  $N$  time periods. A problem of interest in this case, is to determine the best possible times of these transmissions so as to minimize the average distortion between the transmitted and received signals.

### III. ESTIMATING AN I.I.D. RANDOM SEQUENCE WITH LIMITED MEASUREMENTS

#### A. Problem Definition

Consider the special case of the general problem defined in Section II, where the source outputs a zero-mean<sup>4</sup> i.i.d. random sequence  $b_k$ ,  $0 \leq k \leq N-1$ . Let  $\mathcal{B}$  denote the range of the random variable  $b_k$ . We assume that  $b_k$ 's have a finite second moment,  $\sigma_b^2 < \infty$ , but their probability distribution remains unspecified for now. At time  $k$ , the encoder/observer makes a sequential measurement of  $b_k$ , and determines whether to access the channel for transmission, which it can only do a limited,  $M \leq N$ , number of times. The channel is noiseless and thus has a capacity to transmit the source output error-free when it is used to transmit. Note that, even when it decides not to use the channel for transmission, the observer/encoder may still convey a 1-bit information to the estimator/decoder. In view of this, the channel input  $x_k$  belongs to the set  $\mathcal{X} := \mathcal{B} \cup \{\text{NT}\}$ , where NT stands for “no transmission.”

More precisely, we let  $s_k$  denote the number of channel uses (or transmissions) left at time  $k$ . Now if  $s_k \geq 1$ , we have  $y_k = x_k$  for  $x_k \in \mathcal{B} \cup \{\text{NT}\}$ . If  $s_k = 0$ , on the other hand, the channel is useless, since

<sup>4</sup>This is not restrictive, as the known mean can be subtracted out by the estimator.

we have exhausted the allocated number of channel uses. Note that, when the channel is noiseless, both the encoder and the decoder can keep track of  $s_k$  by initializing  $s_0 = M$  and decrementing it by 1 every time a transmission decision is taken.

We want to design an estimator/decoder

$$\hat{b}_k = \hat{\mu}_k(I_k^d) \text{ for } 0 \leq k \leq N-1$$

based on the available information  $I_k^d$  at time  $k$ . Clearly, the information available to the estimator is controlled by the observer. The average distortion between the observed and estimated processes can be taken to be the average mean square error as given by (1), or the probability of error distortion measure which is given by (2).

The information  $I_k^d$  available to the estimator at time  $k$  is a result of an outcome of decisions taken by the observer up until time  $k$ . Let the observer's decision at time  $k$  be

$$x_k = \mu_k(I_k^e)$$

where  $I_k^e$  is the information available to the observer at time  $k$ . Assuming perfect recall, we have

$$\begin{aligned} I_0^e &= \{(s_0, t_0); b_0\} \\ I_k^e &= \{(s_k, t_k); b_0^k; x_0^{k-1}\}, \quad 1 \leq k \leq N-1 \end{aligned}$$

where  $t_k$  denotes the number of time, or decision slots left at time  $k$ . We have

$$t_{k+1} = t_k - 1, \quad 0 \leq k \leq N-2$$

with  $t_0 = N$ .

The range of  $\mu_k(\cdot)$  is the space  $\mathcal{X} = \mathcal{B} \cup \{\text{NT}\}$ . Let  $\sigma_k$  denote the decision whether the observer has decided to transmit or not. Assume  $s_k \geq 1$ , and let  $\sigma_k = 1$  if a transmission takes place; i.e.,  $x_k \in \mathcal{B}$ , and  $\sigma_k = 0$  if no transmission takes place. We have

$$s_{k+1} = s_k - \sigma_k, \quad 0 \leq k \leq N-2$$

with  $s_0 = M$ .

The observer's decision at time  $k$  is a function of its  $k$  past measurements, and  $k-1$  past decisions, i.e.,

$$\mu_k(I_k^e) : \mathcal{B}^k \times \mathcal{X}^{k-1} \rightarrow \mathcal{X}, \quad 0 \leq k \leq N-1$$

Now, the information  $I_k^d$  available to the estimator at time  $k$  can be written as

$$I_k^d = \{(s_k, t_k); y_0^k\}, \quad 0 \leq k \leq N-1$$

By definition, the channel output  $y_k$  satisfies  $y_k = x_k$  if  $s_k \geq 1$ , and  $y_k \in \emptyset$  (i.e., no information) if  $s_k = 0$ .

Consider the class of observer-estimator (encoder-decoder) policies consisting of a sequence of functions

$$\Pi = \{\mu_0, \hat{\mu}_0, \dots, \mu_{N-1}, \hat{\mu}_{N-1}\}$$

where each function  $\mu_k$  maps  $I_k^e$  into  $\mathcal{X}$ , and  $\hat{\mu}_k$  maps  $I_k^d$  into  $\mathcal{B}$ ,<sup>5</sup> with the additional restriction that  $\mu_k$  can map to  $\mathcal{B}$  at most  $M$  times. Such policies are called *admissible*.

We want to find an admissible policy  $\pi^* \in \Pi$  that minimizes the average  $N$ -stage distortion, or estimation error:

$$e_{(M,N)}^\pi = E \left\{ \sum_{k=0}^{N-1} (b_k - \hat{\mu}_k(I_k^d))^2 \right\} \quad (3)$$

or for source processes,  $b_k$ , with discrete probability densities:

$$e_{(M,N)}^\pi = E \left\{ \sum_{k=0}^{N-1} \mathcal{I}_{b_k \neq \hat{\mu}_k(I_k^d)} \right\} \quad (4)$$

That is

$$e_{(M,N)}^* = \min_{\pi \in \Pi} e_{(M,N)}^\pi$$

Note that, we omitted the factor of  $\frac{1}{N}$  from the average error expressions for convenience.

If  $M \geq N$ , this problem has the trivial solution where the observer writes the source output  $b_k$  directly into the channel at each time  $k$  (i.e.,  $\mu_k^*(b_k) = b_k$ ), and since the channel is noiseless, the estimator can use an identity mapping (i.e.,  $\hat{\mu}_k^*(I_k^d) = b_k$ ), resulting in zero distortion. Therefore, we only consider the case when  $M < N$ .

Before closing our account on this section, we would like to note the nonclassical nature of the information in this problem. Clearly, the observer's action affects the information available to the estimator, and there is no way in which the estimator can infer the information available to the observer. Also note the order of actions between the decision makers in the problem. At time  $k$ , first the random variable  $b_k$  becomes available, then the observer acts by transmitting some data or not, and finally, the estimator acts by estimating the state with  $\hat{\mu}_k$ , the cost is incurred, and we move to the next time  $k+1$ .

## B. Structure of the Solution

We first consider the problem of finding the optimal estimator  $\hat{\mu}_k^*$  at time  $k$ . Note that the estimator  $\hat{\mu}_k$  appears only in a single term in the error expressions (3)-(4). Thus, for the mean-square error criterion,

<sup>5</sup>Note that we do not distinguish between the source and user sets.

the optimal estimator is simply the solution of the quadratic minimization problem

$$\min_{\hat{\mu}_k(I_k^d)} E \left\{ (b_k - \hat{\mu}_k(I_k^d))^2 | I_k^d \right\}$$

which is given by the conditional expectation of  $b_k$  given the available information at time  $k$ :

$$\hat{\mu}_k^*(I_k^d) = E \{ b_k | I_k^d \} = E \{ b_k | (s_k, t_k); y_0^k \} \quad (5)$$

Similarly, for the probability of error distortion criterion, the optimal estimator is the solution of the minimization problem

$$\min_{\hat{\mu}_k(I_k^d)} E \left\{ \mathcal{J}_{b_k \neq \hat{\mu}_k(I_k^d)} | I_k^d \right\}$$

If at time  $k$  the channel can still be used ( $s_k \geq 1$ ), the solution to this problem is given by the maximum *a posteriori* probability (MAP) estimate of the random variable  $b_k$  given the available information at time  $k$ :

$$\hat{\mu}_k^*(I_k^d) = \arg \max_{m_i \in \mathcal{B}_k(I_k^d)} \delta(y_k - i) p_i = \arg \max_{m_i \in \mathcal{B}_k((s_k, t_k); y_0^k)} p_i \quad (6)$$

where  $\mathcal{B}_k(I_k^d) \subset \mathcal{B}$  is some subset of the range of the random variable  $b_k$ , which we assume is countable. Let  $m_i$  denote the values the random variable  $b_k$  takes. Then,  $p_i$ 's denote the probability mass function of the random variable  $b_k$ , i.e.,  $p_i = P[b_k = m_i]$ .

Note that, for the probability of error distortion criterion, if the channel is useless at time  $k$  (i.e.,  $s_k = 0$ ), the best estimate of  $b_k$  is simply given by

$$\hat{\mu}_k^*(I_k^d) = \arg \max_{m_i \in \mathcal{B}} p_i \quad (7)$$

since the past channel outputs,  $y_0^{k-1}$ , are independent of  $b_k$ .

Similarly, for the mean-square error criterion, the channel output  $y_k$  has no information on  $b_k$  if  $s_k = 0$ . Thus, in this case, the conditional expectation in (5) equals

$$\hat{\mu}_k^*(I_k^d) = E \{ b_k | (0, t_k); y_0^{k-1}, y_k \} = E \{ b_k \} = 0 \quad (8)$$

since again the past channel outputs,  $y_0^{k-1}$ , are generated by the  $\sigma$ -algebra of random variables  $b_0^{k-1}$ , and hence are independent from  $b_k$ .

If  $s_k \geq 1$ , the channel output  $y_k = x_k$ , but since  $y_0^{k-1} = x_0^{k-1}$  is the outcome of a Borel-measurable function defined on the  $\sigma$ -algebra generated by  $b_0^{k-1}$ , the conditional expectation in (5) is equivalent to

$$\hat{\mu}_k^*(I_k^d) = E \{ b_k | (s_k, t_k); x_k \} \quad (9)$$

By a similar argument we can write (6) as

$$\hat{\mu}_k^*(I_k^d) = \arg \max_{m_i \in \mathcal{B}_k((s_k, t_k); x_k)} p_i \quad (10)$$

Now, substituting the optimal estimators (9)-(10) back into the estimation error expressions (3)-(4) yields

$$e_{(M,N)}^\pi = E \left\{ \sum_{k=0}^{N-1} (b_k - E\{b_k | (s_k, t_k); x_k\})^2 \right\} \quad (11)$$

and

$$e_{(M,N)}^\pi = E \left\{ \sum_{k=0}^{N-1} \mathcal{J}_{b_k \neq \arg \max_{m_i \in \mathcal{B}_k((s_k, t_k); x_k)} p_i} \right\} \quad (12)$$

which we seek to minimize over the observer/encoder policies  $\mu_k(I_k^e), 0 \leq k \leq N-1$ . Since  $x_k = \mu_k(I_k^e)$ , we see that the choice of an observer policy affects the cost only through the information made available to the estimator.

In general, the observer's decision  $\mu_k$  at time  $k$  depends on  $(s_k, t_k)$ , all past measurements  $b_0^{k-1}$ , the present measurement  $b_k$ , and its past actions  $x_0^{k-1}$ . However, as we show next, there is nothing the observer can gain by having access to its past measurements  $b_0^{k-1}$  and its past actions  $x_0^{k-1}$  as far as the optimization of the criteria (11)-(12) are concerned. Thus, a *sufficient statistics* for the observer are the current measurement  $b_k$  and the remaining number of channel uses (transmission opportunities) and decision instances, i.e.  $(s_k, t_k)$ .

*Proposition 1:* The set  $S_k^e = \{(s_k, t_k); b_k\}$  constitutes sufficient statistics  $S_k^e(I_k^e)$  for the optimal policy  $\mu_k^*$  of the observer. In other words,

$$\mu_k^*(I_k^e) = \bar{\mu}(S_k^e(I_k^e))$$

for some function  $\bar{\mu}$ .

*Proof:* Suppose we would like to determine the optimal observer policy  $\mu_k^*(I_k^e)$  at time  $k$ , where  $0 \leq k \leq N-1$  is arbitrary. Due to the sequential nature of the decision problem, any observer policy we decide on at time  $k$  will only affect the error  $e_k$  incurred after time  $k$ , i.e.,<sup>6</sup>

$$e_k = E \left\{ \sum_{n=k}^{N-1} (b_n - E\{b_n | (s_n, t_n); x_n\})^2 \right\}$$

Taking the conditional expectation given the available information  $I_k^e$ , under any observer policy  $\mu_k(I_k^e)$  we have

$$E\{e_k | (s_k, t_k); b_0^k; x_0^{k-1}\} = E\{e_k | (s_k, t_k); b_k\}$$

<sup>6</sup>Here, we give the proof only for the error criterion (11). An identical proof can be constructed for the probability of error distortion criterion (12).

because  $b_k^{N-1}$  is independent of  $b_0^{k-1}$ , and  $x_0^{k-1}$  is the outcome of a Borel-measurable function defined on the  $\sigma$ -algebra generated by  $b_0^{k-1}$ . Hence, at time  $k$ , the knowledge of  $b_0^{k-1}$  and  $x_0^{k-1}$  is redundant. ■

A consequence of Proposition 1 is that the observer's decision to use the channel to transmit a source measurement or not is based purely on the current observation  $b_k$  and its past actions only through  $(s_k, t_k)$ .

Since  $\mu_k$  depends explicitly only on the current source output  $b_k$ , the search for an optimal observer policy can be narrowed down to the class of policies of the form<sup>7</sup>

$$\mu_k(I_k^e) = \bar{\mu}((s_k, t_k); b_k) = \begin{cases} b_k & \text{if } b_k \in \mathcal{T}_{(s_k, t_k)} \\ \text{NT} & \text{if } b_k \in \mathcal{T}_{(s_k, t_k)}^c \end{cases} \quad (13)$$

where  $\mathcal{T}_{(s_k, t_k)}$  is a measurable set on  $\mathcal{B}$  and is a function of  $(s_k, t_k)$ . The complement of the set  $\mathcal{T}_{(s_k, t_k)}$  is taken with respect to  $\mathcal{B}$ , i.e.,  $\mathcal{T}_{(s_k, t_k)}^c = \mathcal{B} \setminus \mathcal{T}_{(s_k, t_k)}$ . When probability of error distortion criterion is used, Proposition 1 implies that  $\mathcal{B}_k((s_k, t_k); \text{NT}) = \mathcal{T}_{(s_k, t_k)}^c$ , and  $\mathcal{B}_k((s_k, t_k); m_i) = m_i$ .

Note that the optimal estimators (9) and (10) have access to  $(s_k, t_k)$  as well. Thus, even when the observer chooses not to transmit  $b_k$ , it can still pass a 1-bit information about  $b_k$  to the estimator provided that  $s_k \geq 1$ . If  $k$  is such that all  $M$  transmissions are concluded prior to time  $k$  (i.e.,  $s_k = 0$ ), the estimators are given by (7)-(8), irrespective of  $b_k$ .

Now, observe that the optimization over the observer policies is equivalent to optimization over the sets  $\mathcal{T}_{(s_k, t_k)}$  for all  $k$  such that

$$\max\{0, M - k\} \leq s_k \leq \min\{t_k, M\}$$

and  $t_k = N - k$ . The nonnegativity of  $s_k$  is a result of the limited channel use constraint. Note that if  $s_{k_0} = 0$  for some  $k_0$ , then  $s_k = 0$  for all  $k$  such that  $k_0 \leq k \leq N - 1$ . At the other extreme, we must have  $s_k \leq N - k$ , since if  $s_k = N - k$ , this means there are as many channel uses left as there are decision instances, and the optimal observer and estimator policies in this case are obvious.

### C. The Solution with the Mean-Square Error Criterion

Let  $(s_k, t_k) = (s, t)$ , and  $e_{(s, t)}^*$  denote the optimal value of the estimation error (or distortion) (11) when the decision horizon is of length  $t$ , and the observer is limited to  $s$  channel uses, where  $s \leq t$ . We know that at time  $k$ , the optimal observation policy will be of the form (13).

<sup>7</sup>As long as  $k$  is such that all  $M$  measurements are not exhausted, i.e.,  $s_k \geq 1$ .

Now, at time  $k + 1$ , depending on the realization of the random variable  $b_k$ , the remaining  $(t - 1)$ -stage estimation error is either  $e_{(s-1,t-1)}^*$ , or  $e_{(s,t-1)}^*$ . Thus, inductively by the DP equation [5], we can write<sup>8</sup>

$$e_{(s,t)}^* = \min_{\mathcal{T}_{(s,t)}} \left\{ e_{(s-1,t-1)}^* \int_{b \in \mathcal{T}_{(s,t)}} f(b) db + e_{(s,t-1)}^* \int_{b \in \mathcal{T}_{(s,t)}^c} f(b) db + \int_{b \in \mathcal{T}_{(s,t)}^c} \left[ b - E\{b | b \in \mathcal{T}_{(s,t)}^c\} \right]^2 f(b) db \right\}$$

where  $f(b)$  is the pdf of the random variables  $b_k$ . If  $b_k$ 's are discrete random variables with a probability mass function (pmf), one has to replace the integrals in the above expression with sums. Expanding out the expectation yields

$$e_{(s,t)}^* = \min_{\mathcal{T}_{(s,t)}} \left\{ e_{(s-1,t-1)}^* \int_{b \in \mathcal{T}_{(s,t)}} f(b) db + e_{(s,t-1)}^* \int_{b \in \mathcal{T}_{(s,t)}^c} f(b) db + \int_{b \in \mathcal{T}_{(s,t)}^c} \left[ b - \frac{\int_{b \in \mathcal{T}_{(s,t)}^c} b f(b) db}{\int_{b \in \mathcal{T}_{(s,t)}^c} f(b) db} \right]^2 f(b) db \right\} \quad (14)$$

To solve for  $e_{(s,t)}^*$ , we first note the boundary conditions  $e_{(t,t)}^* = 0$ , and  $e_{(0,t)}^* = t\sigma_b^2$ ,  $\forall t \geq 0$ , where  $\sigma_b^2$  is the variance of  $b_k$ . The term  $e_{(s,t)}^*$  remains undefined for  $s > t$ . The optimal sets satisfy the boundary conditions  $\mathcal{T}_{(t,t)}^* = \mathcal{B}$ , and  $\mathcal{T}_{(0,t)}^* = \emptyset$ ,  $\forall t \geq 0$ . The recursion of (14) needs to be solved offline and the optimal sets  $\mathcal{T}_{(s,t)}^*$  must be tabulated starting with smaller values of  $(s, t)$ .<sup>9</sup> The solution to the original problem can then be determined as follows:

Initialize  $s_0 = M$ ,  $t_0 = N$ . For each  $k$  in  $0 \leq k \leq N - 1$  do the following:

- 1) Look up the optimal set  $\mathcal{T}_{(s_k, t_k)}^*$  from the table that was determined offline.
- 2) Observe  $b_k$ , and apply the observation policy

$$\bar{\mu}^*((s_k, t_k); b_k) = \begin{cases} b_k & \text{if } b_k \in \mathcal{T}_{(s_k, t_k)}^* \\ \text{NT} & \text{if } b_k \in \mathcal{T}_{(s_k, t_k)}^{*c} \end{cases}$$

- 3) Apply the estimation policy

$$\hat{\mu}_k^*(\mathcal{T}_{(s_k, t_k)}^*) = E\{b_k | b_k \in \mathcal{T}_{(s_k, t_k)}^{*c}\} = \frac{\int_{b \in \mathcal{T}_{(s_k, t_k)}^{*c}} b f(b) db}{\int_{b \in \mathcal{T}_{(s_k, t_k)}^{*c}} f(b) db}$$

- 4) Update

$$s_{k+1} = s_k - \sigma_k, \quad t_{k+1} = t_k - 1$$

<sup>8</sup>Assuming that the random variables  $\{b_k\}$  are continuous with a well-defined probability density function (pdf)  $f(b)$ .

<sup>9</sup>Note that (1,2) is the smallest possible nontrivial value.

In tabulating  $\mathcal{T}_{(s,t)}^*$  one should start with solving for  $\mathcal{T}_{(1,2)}^*$ , and the corresponding estimation error  $e_{(1,2)}^*$ . To determine the optimal set at  $(s,t)$ , we need to know the optimal costs at  $(s,t-1)$ , and  $(s-1,t-1)$ . Hence, we can propagate our calculations as shown in Figure 3 starting with  $(s,t) = (1,2)$ .

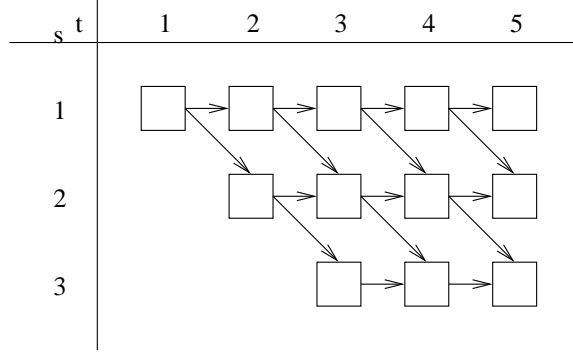


Fig. 3. Recursive calculation of  $e_{(s,t)}^*$ .

Now, we come back to the problem of minimizing (14) over  $\mathcal{T}_{(s,t)}$ . Expanding out the expression inside the minimization we get

$$e_{(s,t)}^* = e_{(s-1,t-1)}^* + \min_{\mathcal{T}_{(s,t)}^c} \left\{ - \left( e_{(s-1,t-1)}^* - e_{(s,t-1)}^* \right) \int_{b \in \mathcal{T}_{(s,t)}^c} f(b) db + \int_{b \in \mathcal{T}_{(s,t)}^c} b^2 f(b) db - \frac{\left[ \int_{b \in \mathcal{T}_{(s,t)}^c} b f(b) db \right]^2}{\int_{b \in \mathcal{T}_{(s,t)}^c} f(b) db} \right\} \quad (15)$$

where we used the fact that  $\int_{b \in \mathcal{T}_{(s,t)}} f(b) db = 1 - \int_{b \in \mathcal{T}_{(s,t)}^c} f(b) db$ .

This an optimization problem over measurable sets  $\mathcal{T}_{(s,t)}^c$  on the real line, and since these sets are not countable, there is no known method for carrying out this minimization in a systematic manner. Therefore, we restrict our search to the sets that are in the form of simple symmetric intervals, i.e.,  $\mathcal{T}_{(s,t)}^c = [-\beta_{(s,t)}, \beta_{(s,t)}]$ , where  $0 \leq \beta_{(s,t)} \leq \infty$ .

Now, because of symmetry, the last term on the right-hand side of (15) disappears from the minimization. Differentiating the remaining terms inside the curly brackets, we obtain the first-order necessary condition:

$$- \left( e_{(s-1,t-1)}^* - e_{(s,t-1)}^* \right) f(\beta_{(s,t)}) + \beta_{(s,t)}^2 f(\beta_{(s,t)}) = 0$$

From which the critical point  $\beta_{(s,t)}^*$  can be determined as<sup>10</sup>

$$\beta_{(s,t)}^* = \sqrt{e_{(s-1,t-1)}^* - e_{(s,t-1)}^*} \quad (16)$$

Note that, we always have  $e_{(s,t-1)}^* \leq e_{(s-1,t-1)}^*$ , since for the same decision horizon,  $t-1$ , the minimum average distortion achieved by  $s$  channel uses, is always less than that achieved by  $s-1$  channel uses. So,  $\beta_{(s,t)}^*$  always exists.

From the first-order condition, we observe that the objective function is strictly decreasing on the interval  $[0, \beta_{(s,t)}^*)$ , and it is strictly increasing on the interval  $(\beta_{(s,t)}^*, \infty)$ . Thus,  $\beta_{(s,t)}^*$  must be a strict global minimizer. Thus, in the class of symmetric intervals, the best set  $\mathcal{T}_{(s,t)}^c$  is given by the interval

$$\mathcal{T}_{(s,t)}^{*c} = [-\sqrt{e_{(s-1,t-1)}^* - e_{(s,t-1)}^*}, \sqrt{e_{(s-1,t-1)}^* - e_{(s,t-1)}^*}] \quad (17)$$

#### D. The Solution with the Probability of Error Criterion

As in Section III-C, let  $(s_k, t_k) = (s, t)$ , and let  $e_{(s,t)}^*$  denote the optimal value of the estimation error (or distortion) (12) when the decision horizon is of length  $t$ , and the observer is limited to  $s$  channel uses, where  $s \leq t$ . We know that at time  $k$ , the optimal observation (transmission) policy will be of the form (13).

Now, at time  $k+1$ , depending on the realization of the random variable  $b_k$ , the remaining  $(t-1)$ -stage estimation error is either  $e_{(s-1,t-1)}^*$ , or  $e_{(s,t-1)}^*$ . Thus, assuming that  $s \geq 1$ , inductively by the DP equation, we can write

$$e_{(s,t)}^* = \min_{\mathcal{T}_{(s,t)}} \left\{ P[b_k \in \mathcal{T}_{(s,t)}] e_{(s-1,t-1)}^* + P[b_k \in \mathcal{T}_{(s,t)}^c] e_{(s,t-1)}^* + P[b_k \in \mathcal{T}_{(s,t)}^c] - \max_{m_j \in \mathcal{T}_{(s,t)}^c} p_j \right\}$$

or equivalently

$$e_{(s,t)}^* = \min_{\mathcal{T}_{(s,t)}^c} \left\{ \left( 1 - P[b_k \in \mathcal{T}_{(s,t)}^c] \right) e_{(s-1,t-1)}^* + P[b_k \in \mathcal{T}_{(s,t)}^c] e_{(s,t-1)}^* + P[b_k \in \mathcal{T}_{(s,t)}^c] - \max_{m_j \in \mathcal{T}_{(s,t)}^c} p_j \right\}$$

Plugging in  $P[b_k \in \mathcal{T}_{(s,t)}^c] = \sum_{m_i \in \mathcal{T}_{(s,t)}^c} p_i$ , and rearranging the terms, we obtain the following error recursion:

$$e_{(s,t)}^* = e_{(s-1,t-1)}^* + \min_{\mathcal{T}_{(s,t)}^c} \left\{ -(e_{(s-1,t-1)}^* - e_{(s,t-1)}^*) \sum_{m_i \in \mathcal{T}_{(s,t)}^c} p_i + \sum_{m_i \in \mathcal{T}_{(s,t)}^c} p_i - \max_{m_j \in \mathcal{T}_{(s,t)}^c} p_j \right\} \quad (18)$$

We next show that the error difference,  $e_{(s-1,t-1)}^* - e_{(s,t-1)}^*$ , can be bounded from below and above.

*Proposition 2:* Suppose  $1 \leq s \leq t$ . Then, the error difference  $e_{(s-1,t-1)}^* - e_{(s,t-1)}^*$  satisfies:

$$0 \leq e_{(s-1,t-1)}^* - e_{(s,t-1)}^* \leq 1$$

<sup>10</sup>The other critical point, namely  $\beta_{(s,t)} = +\infty$ , yields a larger cost.

*Proof:* The lower bound can be established by observing that for the same decision horizon,  $t - 1$ , the minimum average distortion achieved by  $s$  channel uses, is always at least as small as the one that can be achieved by  $s - 1$  channel uses. For the upper bound, one needs to observe that the maximum stage-wise estimation error is bounded by 1.  $\blacksquare$

Using Propostion 2, we will next show that the optimum choice for the sets  $\mathcal{T}_{(s,t)}^c$  is the singleton  $\mathcal{T}_{(s,t)}^{c*} = \{m_{i^*}\}$ , where  $i^* = \arg \max_{m_i \in \mathcal{B}} p_i$ .

In other words, the optimal solution is not to transmit the *most likely* outcome, and transmit all the other outcomes of the source process  $b_k$ . Moreover, this policy is independent of the number of decision instances left,  $t_k$ , and the number of transmission opportunities left,  $s_k$ , provided that  $s_k \geq 1$ . Recall that, the optimum estimator is the MAP estimator and is given by (10).

In order to show that this is indeed the optimal observer (or transmission) policy, we first set the cardinality of the set  $\mathcal{T}_{(s,t)}^c$  to  $|\mathcal{T}_{(s,t)}^c| = 0$ , and determine that the expression inside the curly brackets in (18) is just 0.

We next set  $|\mathcal{T}_{(s,t)}^c| = 1$ , and note that the minimization of the function inside the curly brackets in (18) is equivalent to the following minimization:

$$\min_i \left\{ (1 - p_i) e_{(s-1,t-1)}^* + p_i e_{(s,t-1)}^* + p_i - p_i \right\}$$

Here  $i$  is such that  $m_i \in \mathcal{B}$ . Canceling  $p_i$ 's and rearranging, we obtain an equivalent minimization problem:

$$\min_i -p_i (e_{(s-1,t-1)}^* - e_{(s,t-1)}^*)$$

By Proposition 2, the error difference,  $e_{(s-1,t-1)}^* - e_{(s,t-1)}^*$ , is nonnegative; thus, the minimum is achieved by picking  $i$  as

$$i^* = \arg \max_{m_i \in \mathcal{B}} p_i$$

This choice yields a minimum value of  $-(e_{(s-1,t-1)}^* - e_{(s,t-1)}^*) p_{i^*}$ . Note that this value is at least as good as the value we obtained when we set the cardinality of the set  $|\mathcal{T}_{(s,t)}^c| = 0$ . Thus, we never pick  $\mathcal{T}_{(s,t)}^c$  such that it has zero cardinality.

Finally, we let  $|\mathcal{T}_{(s,t)}^c| \geq 2$ , and let  $p_{\max}$  denote the element of  $\mathcal{T}_{(s,t)}^c$  with the maximal probability. That is,

$$p_{\max} = \max_{m_j \in \mathcal{T}_{(s,t)}^c} p_j$$

Since the number of elements of  $\mathcal{T}_{(s,t)}^c$  is at least 2, the minimization problem inside the curly brackets

in (18) can be written as

$$\min_{\mathcal{T}_{(s,t)}^c} \left\{ -(e_{(s-1,t-1)}^* - e_{(s,t-1)}^*)p_{\max} + (1 - (e_{(s-1,t-1)}^* - e_{(s,t-1)}^*)) \sum_{m_i \in \mathcal{T}_{(s,t)}^c \setminus m_{j^*}} p_i \right\}$$

where

$$m_{j^*} = \arg \max_{m_i \in \mathcal{T}_{(s,t)}^c} p_i$$

Now, by Proposition 2, the term multiplying the sum  $\sum_{m_i \in \mathcal{T}_{(s,t)}^c \setminus m_{j^*}} p_i$  is always nonnegative; hence, we can conclude that the above minimum is bounded from below by

$$-(e_{(s-1,t-1)}^* - e_{(s,t-1)}^*)p_{\max}$$

for any choice of the set  $\mathcal{T}_{(s,t)}^c$  with cardinality  $|\mathcal{T}_{(s,t)}^c| \geq 2$ . However the expression  $-(e_{(s-1,t-1)}^* - e_{(s,t-1)}^*)p_{\max}$  satisfies

$$-(e_{(s-1,t-1)}^* - e_{(s,t-1)}^*)p_{\max} \geq -(e_{(s-1,t-1)}^* - e_{(s,t-1)}^*)p_{i^*}$$

since  $p_{i^*} \geq p_{\max}$ . Therefore, the minimum value of the function inside the curly brackets in (18) is achieved when  $\mathcal{T}_{(s,t)}^c = \{m_{i^*}\}$ , as claimed.

In summary, when the distortion criterion is the probability of error, at time  $k$ , the optimal observer first observes the source output  $b_k$ . Then, it checks to see if  $s_k \geq 1$ ; if so, it transmits  $b_k$  unless  $b_k = m_{i^*}$ , i.e., the most likely outcome. The estimator (or decoder), on the other hand, employs the MAP estimation rule given the output of the channel.

### E. Gaussian Case

We now come back to the mean-square error distortion criterion, and study the case when the source is Gaussian in more detail. Suppose  $b_k$ 's are zero-mean, i.i.d. Gaussian with the pdf

$$f(b) = \frac{1}{\sqrt{2\pi\sigma_b^2}} e^{-\frac{b^2}{2\sigma_b^2}} \quad (19)$$

Let  $\Phi(\cdot)$  denote the cumulative density function (CDF) of the standard Gaussian random variable with zero mean and unit variance. We list some properties of the Gaussian pdf here for completeness, as the derivations that follow use these expressions extensively.

•

$$\int_a^b f(x)dx = \Phi\left(\frac{b}{\sigma_x}\right) - \Phi\left(\frac{a}{\sigma_x}\right)$$

•

$$\int_a^b xf(x)dx = \frac{\sigma_x^2}{\sqrt{2\pi\sigma_x^2}} \left( e^{-\frac{a^2}{2\sigma_x^2}} - e^{-\frac{b^2}{2\sigma_x^2}} \right)$$

•

$$\int_a^b x^2 f(x)dx = \sigma_x^2 \left[ \Phi\left(\frac{b}{\sigma_x}\right) - \Phi\left(\frac{a}{\sigma_x}\right) \right] + \frac{\sigma_x^2}{\sqrt{2\pi\sigma_x^2}} \left( ae^{-\frac{a^2}{2\sigma_x^2}} - be^{-\frac{b^2}{2\sigma_x^2}} \right)$$

In the Gaussian case, we can generalize our search for an optimum in (15) to more general intervals of the form  $\mathcal{T}_{(s,t)}^c = [\alpha_{(s,t)}, \beta_{(s,t)}]$ , where  $-\infty \leq \alpha_{(s,t)} \leq \beta_{(s,t)} < +\infty$ .

Figure 4 shows the plot of the objective function on the right-hand side of (15) for the case when  $\mathcal{T}_{(s,t)} = [a, b]$ ,  $\sigma_b^2 = 1$ ,  $e_{(s-1,t-1)}^* = 3$ , and  $e_{(s,t-1)}^* = 1$ . Note that the minimum occurs at  $b^* = -a^* = \sqrt{3-1} = \sqrt{2} = 1.4142$ . Thus, even though we did not restrict ourselves to symmetric intervals, the solution is still a symmetric interval around zero. To show that this is indeed the case in general, one needs to differentiate the objective function inside the curly brackets in (15) with respect to both  $\alpha_{(s,t)}$  and  $\beta_{(s,t)}$ , and show that the minimum occurs at  $\beta_{(s,t)}^* = -\alpha_{(s,t)}^*$  when  $f(b)$  is the Gaussian pdf [2].

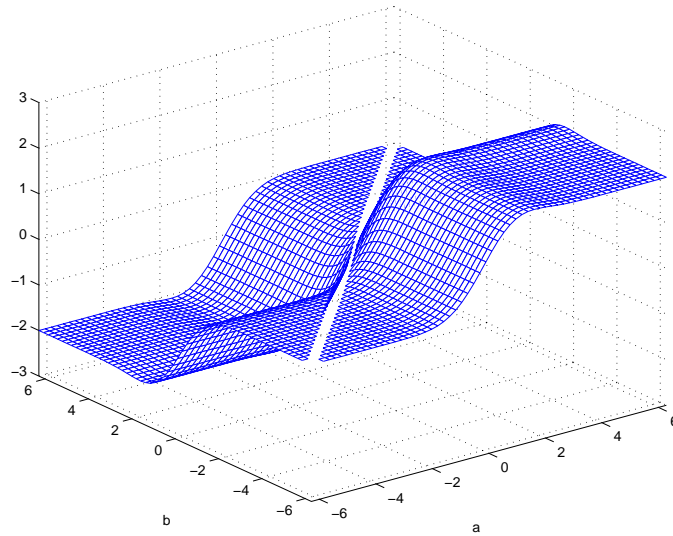


Fig. 4. Plot of the objective function in the Gaussian case with  $\mathcal{T}_{(s,t)} = [a, b]$  when  $\sigma_b^2 = 1$ ,  $e_{(s-1,t-1)}^* = 3$ , and  $e_{(s,t-1)}^* = 1$ .

To evaluate the optimum estimation error  $e_{(s,t)}^*$  in terms of  $e_{(s-1,t-1)}^*$  and  $e_{(s,t-1)}^*$ , we substitute the optimum interval solution (17) into the right-hand side of (15), and use the standard properties of the

Gaussian density that we listed above to obtain

$$e_{(s,t)}^* = e_{(s-1,t-1)}^* - \left[ e_{(s-1,t-1)}^* - e_{(s,t-1)}^* - \sigma_b^2 \right] \left[ 2\Phi \left( \sqrt{\frac{e_{(s-1,t-1)}^* - e_{(s,t-1)}^*}{\sigma_b^2}} \right) - 1 \right] - \frac{2\sigma_b^2}{\sqrt{2\pi\sigma_b^2}} \sqrt{e_{(s-1,t-1)}^* - e_{(s,t-1)}^*} e^{-\frac{e_{(s-1,t-1)}^* - e_{(s,t-1)}^*}{2\sigma_b^2}} \quad (20)$$

We can normalize the optimal estimation error by letting

$$\varepsilon_{(s,t)} = \frac{e_{(s,t)}^*}{\sigma_b^2} \quad (21)$$

and rewrite the recursion (20) in a simpler form:

$$\varepsilon_{(s,t)} = \varepsilon_{(s-1,t-1)} - \left[ \varepsilon_{(s-1,t-1)} - \varepsilon_{(s,t-1)} - 1 \right] \left[ 2\Phi \left( \sqrt{\varepsilon_{(s-1,t-1)} - \varepsilon_{(s,t-1)}} \right) - 1 \right] - \frac{2}{\sqrt{2\pi}} \sqrt{\varepsilon_{(s-1,t-1)} - \varepsilon_{(s,t-1)}} e^{-\frac{\varepsilon_{(s-1,t-1)} - \varepsilon_{(s,t-1)}}{2}} \quad (22)$$

with the initial conditions

$$\varepsilon(t,t) = 0, \quad \varepsilon(0,t) = t, \quad \forall t \geq 0$$

and  $\varepsilon(s,t)$  is undefined for  $s > t$ .

Hence, we can provide a solution to the problem of optimal sequential estimation of an i.i.d. Gaussian process of finite length over a noiseless channel that can only be used a limited number of times. First, a table has to be formed by an offline numerical computation of the recursion (22). Then, the table can be scaled, if needed, to the actual variance of the process through (21). Next, the transmission intervals for the observer are determined via (16), and tabulated for all feasible pairs  $(s,t)$ . For the online computation, as illustrated in Section III-C, the observer has to keep two states,  $(s_k, t_k)$ . Each time unit  $k$ , after observing the realization of the random variable  $b_k$ , the observer compares the realized value of the random variable to the optimum decision interval corresponding to the current state  $(s_k, t_k)$ , and makes a transmission decision. The estimator, on the other hand, has access to the same tabulated values of the transmission intervals,  $\mathcal{T}_{(s,t)}^*$ , and it keeps track of the states  $(s_k, t_k)$  in the same way the observer does. Upon receiving the transmitted data,  $y_k$ , from the channel, the estimator simply applies the estimation policy given in Section III-C.

#### F. Limiting Solution for the Gaussian Case

The normalized recursion of the estimation error in the Gaussian case is given by (22). Note that, this is a Riccati-like equation, which needs to be solved offline to determine the optimal observation and

estimation policies. However, unlike the RE, (22) is a two-dimensional recursion with given boundary conditions. In order to see how this recursion behaves when  $(s, t)$  are large, we proceed as follows. Let  $t = s + k$ , where  $k \in \mathcal{N}$  is an arbitrary integer. We substitute  $s + k$  for  $t$  in the recursion (22) starting with  $k = 1$ , and obtain

$$\varepsilon_{(s,s+1)} = \varepsilon_{(s-1,s)} - [\varepsilon_{(s-1,s)} - 1] \times [2\Phi(\sqrt{\varepsilon_{(s-1,s)}}) - 1] - \frac{2}{\sqrt{2\pi}} \sqrt{\varepsilon_{(s-1,s)}} e^{-\frac{\varepsilon_{(s-1,s)}}{2}} \quad (23)$$

since  $e(s, s) = 0$ . The sequence (23), can be shown to converge from the initial condition  $\varepsilon(0, 1) = 1$  to the limit  $l_\varepsilon \approx 0.3345$ , which is the solution of the nonlinear equation:

$$(1 - l_\varepsilon)(2\Phi(\sqrt{l_\varepsilon}) - 1) = \frac{2}{\sqrt{2\pi}} \sqrt{l_\varepsilon} e^{-\frac{l_\varepsilon}{2}}$$

Proceeding in the same manner, we can calculate the limits of the sequences  $\varepsilon_{(s,s+k)}$  for an arbitrary but otherwise fixed  $k$ , and it turns out that

$$\lim_{s \rightarrow \infty} \varepsilon_{(s,s+k)} = l_\varepsilon k \approx 0.3345k$$

Since  $k$  is arbitrary, we conclude that for  $t \geq s$ ,

$$\lim_{s \rightarrow \infty, t \rightarrow \infty} \varepsilon_{(s,t)} = l_\varepsilon(t - s)$$

As a result, from (17), the optimum interval for the observer not to transmit is given by

$$[-\sqrt{l_\varepsilon}, \sqrt{l_\varepsilon}] \approx [-0.5784, 0.5784] \quad (24)$$

Note that the interval given in (24) is for the normalized Gaussian density. For an arbitrary Gaussian distribution with variance  $\sigma_b^2$ , the interval must be scaled by the standard deviation  $\sigma_b$ , i.e.,

$$[-\sqrt{l_\varepsilon} \sigma_b, \sqrt{l_\varepsilon} \sigma_b]$$

Note that, in the limit as  $(M, N)$  get large, the  $N$ -stage optimal average distortion,  $D_{(M,N)}^* = \frac{1}{N} e_{(M,N)}^*$ , for a Gaussian i.i.d. source with variance  $\sigma_b^2$  is given by

$$\lim_{M \rightarrow \infty, N \rightarrow \infty} D_{(M,N)}^* = l_\varepsilon \frac{1}{N} (N - M) \sigma_b^2 = l_\varepsilon \left(1 - \frac{M}{N}\right) \sigma_b^2$$

One can identify the term  $\left(1 - \frac{M}{N}\right) \sigma_b^2$  as the average distortion when no observer is used. More precisely, say there is no observer agent, and thus the source is allowed to make  $M$  transmissions at some arbitrary times. When the estimator receives the source data, clearly the estimation error for that stage is zero. When no measurement is received on the other hand, the estimator simply estimates the process by its  $a$

*priori* density, and incurs an error of size  $\sigma_b^2$ . This process of estimation leads to an average estimation error of size

$$e_{no-observer} = \left(1 - \frac{M}{N}\right) \sigma_b^2$$

Comparing this error with the asymptotic optimum estimation error obtained by using an optimum observer-estimator (encoder-decoder) pair, we see that they differ by a factor of  $l_\varepsilon$ . Hence,  $(1 - l_\varepsilon)$  can be thought of as the encoder-decoder gain of the communication system in Figure 1 for a Gaussian i.i.d. source and a noiseless channel.

### G. Gaussian Case with Noisy Measurements

Let the source process  $b_k$  be i.i.d. Gaussian. If the observer has access to a noisy version of the source output, i.e.,

$$z_k = b_k + v_k$$

where  $v_k$  is zero-mean, i.i.d. Gaussian<sup>11</sup> with variance  $\sigma_v^2$ , the optimization problem with the mean-square distortion measure can be solved using a similar approach. In this case, the observer's decision as to whether to use the channel to transmit or not depends on the available data  $z_k$ . In the derivation of the optimal observer-estimator pair, most of the analysis of Section III-E carries over.

In order to see that the structure of the solution is preserved, first observe that when  $s_k = 0$ ,  $\hat{\mu}_k^* = 0$ , and for  $s_k \geq 1$ , the optimal estimator has the form:

$$\hat{\mu}_k^*(I_k^d) = E\{b_k | (s_k, t_k); x_k\}$$

Substituting this into the error expression, and following along the lines of Proposition 1, one can see that the optimal observer policy has the form

$$\mu_k(I_k^e) = \bar{\mu}((s_k, t_k); z_k)$$

In other words  $\{(s_k, t_k); z_k\}$  is a sufficient statistics for the optimal policy  $\mu_k^*(I_k^e)$ .

Since  $\mu_k$  depends explicitly only on the current measurement  $z_k$ , for  $s_k \geq 1$ , the search for an optimal encoder policy can be narrowed down to the class of policies of the form

$$\mu_k(I_k^e) = \bar{\mu}((s_k, t_k); z_k) = \begin{cases} z_k & \text{if } z_k \in \mathcal{T}_{(s_k, t_k)} \\ \text{NT} & \text{if } z_k \in \mathcal{T}_{(s_k, t_k)}^c \end{cases}$$

<sup>11</sup>We also assume that the processes  $\{b_k\}$  and  $\{v_k\}$  are independent.

where  $\mathcal{T}_{(s_k, t_k)}$  is a measurable set on  $\mathcal{B}$ , and is a function of  $(s_k, t_k)$ . Since  $x_k = \mu_k(I_k^e)$ , for  $s_k \geq 1$ , we can write the optimal estimator as

$$\hat{\mu}_k((s_k, t_k); x_k) = \begin{cases} \frac{\sigma_b^2}{\sigma_b^2 + \sigma_v^2} z_k & \text{if } z_k \in \mathcal{T}_{(s_k, t_k)} \\ E \left\{ b_k | z_k \in \mathcal{T}_{(s_k, t_k)}^c \right\} & \text{if } z_k \in \mathcal{T}_{(s_k, t_k)}^c \end{cases}$$

We proceed as in Section III-C, and write the dynamic programming recursion governing the evolution of the optimal estimation error as follows:

$$e_{(s,t)}^* = \min_{\mathcal{T}_{(s,t)}^c} \left\{ e_{(s-1,t-1)}^* P[z \in \mathcal{T}_{(s,t)}] + \sigma_b^2 + \left( \frac{\sigma_b^2}{\sigma_b^2 + \sigma_v^2} \right)^2 \int_{z \in \mathcal{T}_{(s,t)}} z^2 f_Z(z) dz + e_{(s,t-1)}^* P[z \in \mathcal{T}_{(s,t)}^c] - 2 \frac{\sigma_b^2}{\sigma_b^2 + \sigma_v^2} \int_{z \in \mathcal{T}_{(s,t)}} z E[b|z] f_Z(z) dz \right\}$$

where  $f_Z(z) \sim N(0, \sigma_b^2 + \sigma_v^2)$ , and  $f_{B|Z}(b|z) \sim N\left(\frac{\sigma_b^2}{\sigma_b^2 + \sigma_v^2} z, \frac{\sigma_b^2 \sigma_v^2}{\sigma_b^2 + \sigma_v^2}\right)$ . The recursion can be simplified as

$$e_{(s,t)}^* = \min_{\mathcal{T}_{(s,t)}^c} \left\{ e_{(s-1,t-1)}^* + \sigma_b^2 - (e_{(s-1,t-1)}^* - e_{(s,t-1)}^*) \int_{z \in \mathcal{T}_{(s,t)}^c} f_Z(z) dz - \left( \frac{\sigma_b^2}{\sigma_b^2 + \sigma_v^2} \right)^2 (\sigma_b^2 + \sigma_v^2) + \left( \frac{\sigma_b^2}{\sigma_b^2 + \sigma_v^2} \right)^2 \int_{z \in \mathcal{T}_{(s,t)}^c} z^2 f_Z(z) dz \right\}$$

Following along the lines of Section III-E, we restrict our search for an optimum set to simple intervals, i.e.,  $\mathcal{T}_{(s,t)}^c = [\alpha_{(s,t)}, \beta_{(s,t)}]$ . The same analysis gives the optimum choice for  $\beta_{(s,t)}$

$$\beta_{(s,t)}^* = \frac{\sigma_b^2 + \sigma_v^2}{\sigma_b^2} \sqrt{e_{(s-1,t-1)}^* - e_{(s,t-1)}^*}$$

and  $\alpha_{(s,t)}^* = -\beta_{(s,t)}^*$ . Substituting these values into the error recursion, we obtain the two-dimensional recursion for the estimation error:

$$e_{(s,t)}^* = e_{(s-1,t-1)}^* - \left[ e_{(s-1,t-1)}^* - e_{(s,t-1)}^* - \frac{(\sigma_b^2)^2}{\sigma_b^2 + \sigma_v^2} \right] \left[ 2\Phi \left( \frac{\sqrt{\sigma_b^2 + \sigma_v^2} \sqrt{e_{(s-1,t-1)}^* - e_{(s,t-1)}^*}}{\sigma_b^2} \right) - 1 \right] + \sigma_b^2 - \frac{(\sigma_b^2)^2}{\sigma_b^2 + \sigma_v^2} - \frac{2\sigma_b^2}{\sqrt{2\pi(\sigma_b^2 + \sigma_v^2)}} \sqrt{e_{(s-1,t-1)}^* - e_{(s,t-1)}^*} e^{-\frac{\left(\frac{\sigma_b^2 + \sigma_v^2}{\sigma_b^2}\right)^2 (e_{(s-1,t-1)}^* - e_{(s,t-1)}^*)}{2(\sigma_b^2 + \sigma_v^2)}}$$

Note that, for  $\sigma_v^2 = 0$  this recursion simplifies to (20), which is the recursion for the perfect state measurements.

We can normalize the optimal estimation error by letting

$$\varepsilon_{(s,t)} = \frac{\sigma_b^2 + \sigma_v^2}{(\sigma_b^2)^2} e_{(s,t)}^* \quad (25)$$

and rewrite the above recursion in a simpler form:

$$\begin{aligned} \varepsilon_{(s,t)} &= \varepsilon_{(s-1,t-1)} - [\varepsilon_{(s-1,t-1)} - \varepsilon_{(s,t-1)} - 1] [2\Phi(\sqrt{\varepsilon_{(s-1,t-1)} - \varepsilon_{(s,t-1)}}) - 1] \\ &\quad + \frac{\sigma_v^2}{\sigma_b^2} - \frac{2}{\sqrt{2\pi}} \sqrt{\varepsilon_{(s-1,t-1)} - \varepsilon_{(s,t-1)}} e^{-\frac{\varepsilon_{(s-1,t-1)} - \varepsilon_{(s,t-1)}}{2}} \end{aligned} \quad (26)$$

with the initial conditions

$$\varepsilon(t,t) = \frac{\sigma_v^2}{\sigma_b^2} t, \quad \varepsilon(0,t) = \left(1 + \frac{\sigma_v^2}{\sigma_b^2}\right) t, \quad \forall t \geq 0$$

and  $\varepsilon(s,t)$  is undefined for  $s > t$ .

We note that the recursion (26) reduces to the recursion (22), as the noise variance  $\sigma_v^2 \rightarrow 0$ .

#### IV. ESTIMATING A GAUSS-MARKOV PROCESS WITH LIMITED MEASUREMENTS

In this section, we discuss the case when the source process is Markov

$$b_{k+1} = Ab_k + w_k$$

driven by an i.i.d. Gaussian process  $\{w_k\}$  with zero-mean. The solution to this case is similar to the Gaussian i.i.d. case when the observer has access to the source output  $b_k$  without noise. The only difference is that, now the observer-estimator pair has to keep track of three variables  $(r_k, s_k, t_k)$ , where  $r_k$  keeps track of the number of time units passed since the last use of the channel for transmission. A similar DP recursion, now in three dimensions, can be obtained.

Let  $r$  denote the number of time units passed since the last transmission of a source output. Reasoning as in Section III-D, we can deduce that for  $s \geq 1$ , the optimal estimator has the form

$$\hat{\mu}((r,s,t); b_{N-t}) = \begin{cases} b_{N-t} & b_{N-t} \in \mathcal{T}_{(r,s,t)} \\ E\{b_{N-t} | b_{N-t} \in \mathcal{T}_{(r,s,t)}^c\} & b_{N-t} \in \mathcal{T}_{(r,s,t)}^c \end{cases}$$

With the estimator structure in place, the error recursion can be derived following along the lines of previous sections:

$$\begin{aligned} e_{(r,s,t)}^* &= \min_{\mathcal{T}_{(r,s,t)}} \left\{ e_{(1,s-1,t-1)}^* P[b_{N-t} \in \mathcal{T}_{(r,s,t)}] + e_{(r+1,s,t-1)}^* P[b_{N-t} \in \mathcal{T}_{(r,s,t)}^c] \right. \\ &\quad \left. + \int_{b_{N-t} \in \mathcal{T}_{(r,s,t)}^c} [b_{N-t} - A^r b_{N-t-r}]^2 f_{b_{N-t}}(b_{N-t}) db_{N-t} \right\} \end{aligned}$$

where  $b_{N-t} \sim N(A^r b_{N-t-r}, (\sum_{k=1}^r A^{2(k-1)}) \sigma_b^2)$ .

Now if we let  $\mathcal{T}_{(r,s,t)}^c = [\alpha_{(r,s,t)}, \beta_{(r,s,t)}]$ , the optimal choices for the parameters  $\alpha_{(r,s,t)}$  and  $\beta_{(r,s,t)}$  are

$$\begin{aligned} \alpha_{(r,s,t)}^* &= A^r b_{N-t-r} + \sqrt{e_{(1,s-1,t-1)}^* - e_{(r+1,s,t-1)}^*} \\ \beta_{(r,s,t)}^* &= A^r b_{N-t-r} - \sqrt{e_{(1,s-1,t-1)}^* - e_{(r+1,s,t-1)}^*} \end{aligned}$$

Substituting these choices back into the error recursion and simplifying yields

$$\begin{aligned}
e_{(r,s,t)}^* &= e^*(1,s-1,t-1) - \left[ e_{(1,s-1,t-1)}^* - e_{(r+1,s,t-1)}^* - \left( \sum_{k=1}^r A^{2(k-1)} \right) \sigma_b^2 \right] \\
&\quad \times \left[ 2\Phi \left( \sqrt{\frac{e_{(1,s-1,t-1)}^* - e_{(r+1,s,t-1)}^*}{\sum_{k=1}^r A^{2(k-1)} \sigma_b^2}} \right) - 1 \right] \\
&\quad - \frac{2\sqrt{\sum_{k=1}^r A^{2(k-1)} \sigma_b^2}}{2\pi} \sqrt{e_{(1,s-1,t-1)}^* - e_{(r+1,s,t-1)}^*} e^{-\frac{e_{(1,s-1,t-1)}^* - e_{(r+1,s,t-1)}^*}{2\sum_{k=1}^r A^{2(k-1)} \sigma_b^2}}
\end{aligned} \tag{27}$$

where we have made use of the fact that for a Gaussian random variable  $x$  with mean  $m$  and variance  $k^2$ , and for  $a \geq 0$ , we have the expression

$$\int_{m-\sqrt{a}}^{m+\sqrt{a}} (x-m)^2 f_x(x) dx = -\frac{2\sqrt{ak}}{\sqrt{2\pi}} e^{-\frac{a}{2k^2}} + k^2 \left( 2\Phi \left( \frac{\sqrt{a}}{k} \right) - 1 \right)$$

The recursion (27) is defined for  $r \geq 1$ , and  $0 \leq s \leq t$  with the boundary conditions given by

$$e_{(r,t)}^* = 0, \quad e_{(r,0,t)}^* = \left( \sum_{l=r}^{r+t-1} \sum_{k=1}^l A^{2(k-1)} \right) \sigma_b^2$$

Note that, as in Section III-E, one can define the normalized estimation error by

$$\mathcal{E}_{(r,s,t)} = \frac{1}{\left( \sum_{k=1}^r A^{2(k-1)} \right) \sigma_b^2} e_{(r,s,t)}^* \tag{28}$$

and simplify the recursion (27) further as follows:

$$\begin{aligned}
\mathcal{E}_{(r,s,t)} &= \mathcal{E}_{(1,s-1,t-1)} - \left[ \mathcal{E}_{(1,s-1,t-1)} - \mathcal{E}_{(r+1,s,t-1)} - 1 \right] \left[ 2\Phi \left( \sqrt{\mathcal{E}_{(1,s-1,t-1)} - \mathcal{E}_{(r+1,s,t-1)}} \right) - 1 \right] \\
&\quad - \frac{2}{\sqrt{2\pi}} \sqrt{\mathcal{E}_{(1,s-1,t-1)} - \mathcal{E}_{(r+1,s,t-1)}} e^{-\frac{\mathcal{E}_{(1,s-1,t-1)} - \mathcal{E}_{(r+1,s,t-1)}}{2}}
\end{aligned}$$

Note that when  $r = 0$ , this is the exact same recursion as in the case of estimating an i.i.d. Gaussian process with no measurement noise. The only difference between this case and the i.i.d. case is in scaling back into the original estimation error via (28). However, unlike the i.i.d. case, this recursion must be solved offline for all feasible  $(r,s,t)$  triplets, and a three-dimensional table has to be formed.

## V. ILLUSTRATIVE EXAMPLES

### A. Example 1

As an example for the case when the source is binary, i.e.,  $b_k \in \{0,1\}$ , consider the problem of sequentially estimating a Bernoulli process of length  $N$  with  $M$  opportunities to transmit over a noiseless binary channel. This problem is a special case of the general problem we solved in Section III-D. The probability distribution of the source is given, and say, without loss of any generality, that 1 is a more likely outcome than 0. In this case, the best observation policy is to start at time  $k = 0$  not transmit the

likely outcome 1, and to use the channel to transmit only the unlikely outcome 0. And the best estimation scheme is to employ the MAP estimator which estimates NT as 1, and 0 as 0, as long as  $s_k \geq 1$ . If  $s_k = 0$ , on the other hand, then the best estimator should estimate 1 regardless of the channel output.

### B. Example 2

The second example is just solving the problem of Section III-E for  $(s, t) = (1, 2)$ . So, the observer can use the channel for transmission only once, at time  $k = 0$  or 1, and the observer and the estimator are jointly trying to minimize the average distortion (or estimation error):

$$e = E \{ (b_0 - \hat{b}_0)^2 + (b_1 - \hat{b}_1)^2 \}$$

where  $b_0, b_1$  are i.i.d. Gaussian with zero mean, and variance  $\sigma_b^2$ . If we arbitrarily choose to transmit the first source output, or the second one, the estimation error would be

$$e_{no-observer}^* = \sigma_b^2$$

which is the best error that can be achieved without a decision maker that observes the source output. Now, suppose the observer is aware of the fact that the estimator knows the *a priori* distribution of  $b_0$ . So, it makes sense for the observer not to transmit the realized value of  $b_0$  if this value happens to be close to the *a priori* estimate of it, which in this case is the mean value of  $b_0$ , i.e., zero.

Motivated by this intuition, the observer decides to adopt a policy in which it will not use the channel to transmit  $b_0$  if it lies in an interval  $[\alpha, \beta]$  around zero. Note that the decision for the second stage would already have been made once  $\alpha$  and  $\beta$  are determined, because, if  $b_0 \in [\alpha, \beta]$ , then the observer cannot use the channel to transmit at time 1, and if  $b_0 \notin [\alpha, \beta]$ , there is no reason why it should not transmit at time 1.

Now, the optimization problem faced by the observer is to choose  $\alpha$  and  $\beta$  such that the following error is minimized:

$$e_{(\alpha, \beta)} = \int_{\alpha}^{\beta} (b - E \{ b | b \in [\alpha, \beta] \})^2 f(b) db + \sigma_b^2 P \{ b_0 \notin [\alpha, \beta] \}$$

where  $f(b)$  is the Gaussian density given by (19). The solution can be easily obtained by checking the first and second order optimality conditions, and is given by

$$(\alpha^*, \beta^*) = (-\sigma_b, \sigma_b)$$

Thus, the observer should not use the channel to transmit the source output  $b_0$  if it falls within one standard deviation of its mean. For these values of  $\alpha$  and  $\beta$ , the optimal value of the estimation error

can be calculated as

$$e_{(\alpha^*, \beta^*)} = \sigma_b^2 \left[ 1 - \sqrt{\frac{2}{\pi e}} \right]$$

Comparing this error to the no-observer policy,  $e_{no-observer}^* = \sigma_b^2$ , we see that there is an approximately  $\sqrt{\frac{2}{\pi e}} \approx 48\%$  improvement in the estimation error.

### C. Example 3

The third and final example we will discuss considers the following design problem. We are given a time-horizon of a fixed length  $N$ , say 100. For this  $N = 100$  time units, we would like to sequentially estimate the state of a zero-mean, i.i.d. Gaussian process with unit variance. We have a design criterion which says that the *aggregate* estimation error should not exceed 20. The solution to this problem without an observer agent is to reveal 80 arbitrary observations to the estimator and achieve an aggregate estimation error of 20. Suppose, now we use the optimal observer-estimator pair. In Figure 5, we plot the optimal value of the 100-stage estimation error for different values of  $M$ .

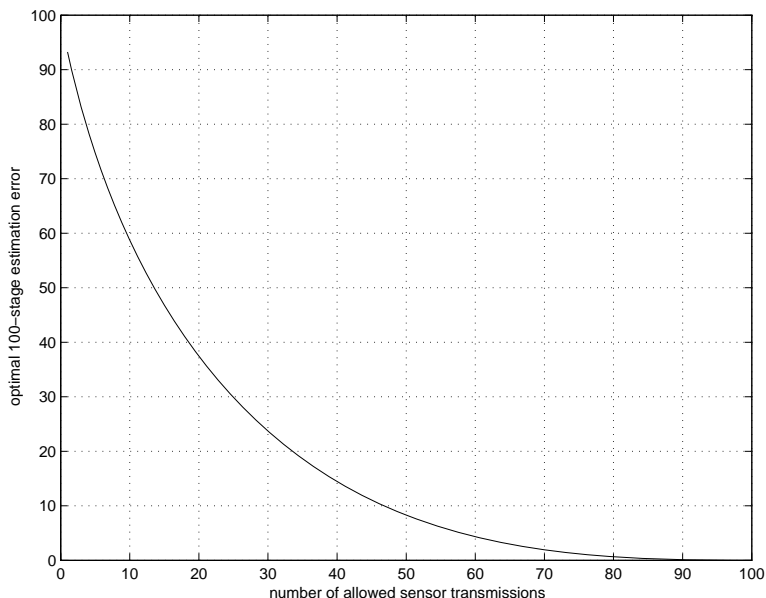


Fig. 5. Optimal 100-stage estimation error vs. the number of allowed channel uses.

It is striking that a cumulative estimation error of 20 can be achieved with only 34 transmissions. This is approximately a  $\frac{80-34}{80} \times 100 \approx 58\%$  improvement over the no-observer policy.

In order to verify our design, we simulate the optimal observer and estimator policies in Matlab. Figures 6 and 7 show typical sample paths of the optimal number of channel uses left for a decision

horizon of length  $N = 100$ , and a limited,  $M = 34$ , number of channel uses. The sample paths depend on the realization of the random sequence  $\{b_k\}_0^{N-1}$ .

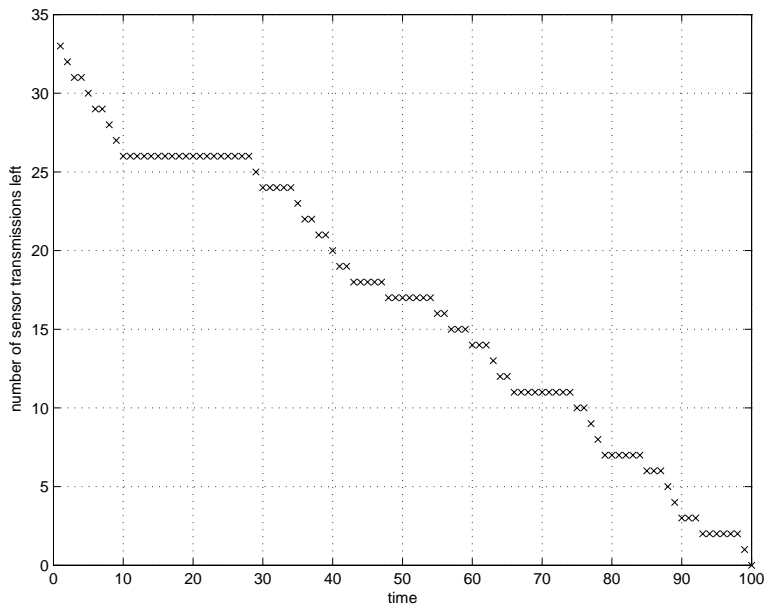


Fig. 6. A typical sample path of the number of channel uses left under the optimal observer-estimator policies.  $(N, M) = (100, 34)$ .

Also, to verify that  $M = 34$  channel uses suffice to guarantee a cumulative average estimation error of 20, we simulate 500 instances of the estimation problem with  $(N, M) = (100, 34)$ . In Figure 8, we plot the realized value of the cumulative estimation error at each run for the optimal observer-estimator case and the no-observer case. Note that the sample path average of the cumulative estimation error (shown as the dotted line) is just below 20, in agreement with our theoretical findings.

## VI. CONCLUSIONS

In this paper, we introduced a new hard-constrained sequential estimation problem with many potential applications. Current research effort is directed towards solving the problem in different settings, and applying the results of this paper to estimation and control problems over wireless sensor networks.

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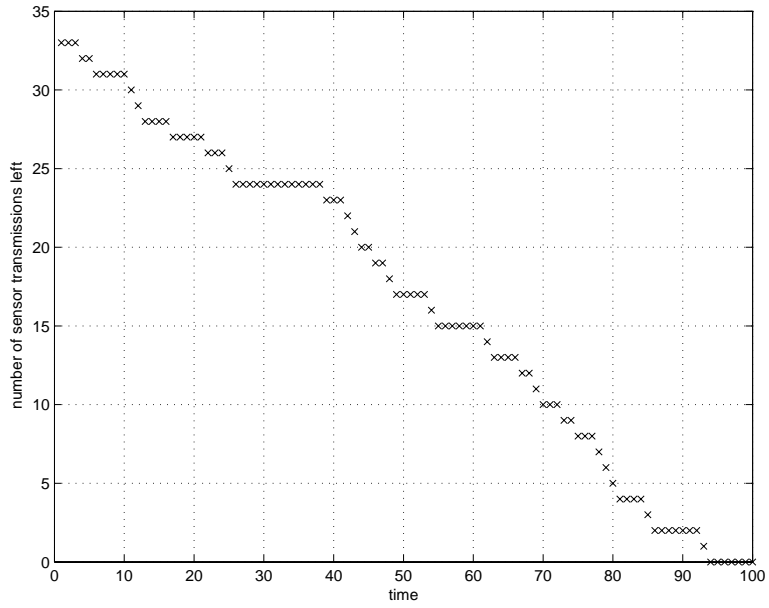


Fig. 7. Another typical sample path of the number of channel uses left under the optimal observer-estimator policies.  $(N, M) = (100, 34)$ .

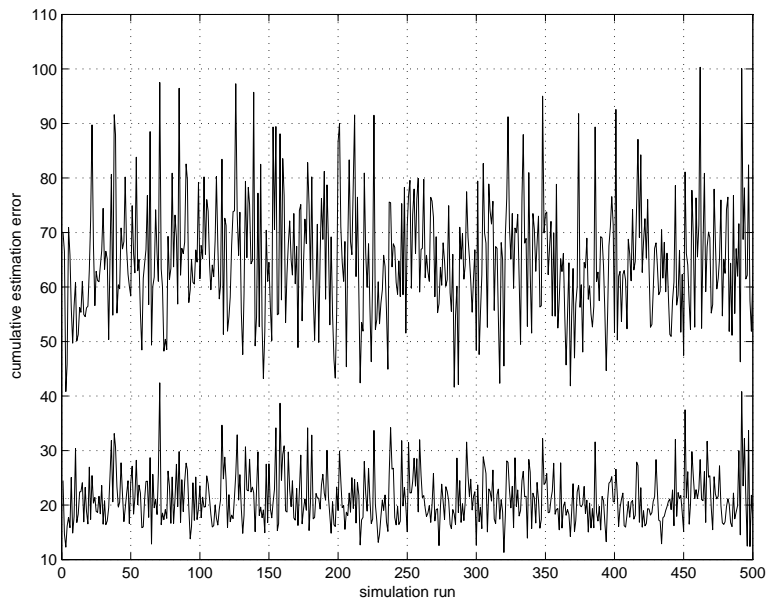


Fig. 8. Comparison of the sample path cumulative estimation errors between the cases when no observer is present, and when the optimal observer-estimator is used.  $(N, M) = (100, 34)$ .

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