Wireless Sensing with Power Constraints

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1 Introduction

Recent advances in wireless technology and standards, such as ZigBee and IEEE 802.15.4, have made wireless sensing solutions feasible for industrial applications [1, 2, 3, 4]. Most of these applications use battery-powered integrated wireless sensing/communication devices, also called motes, for data logging and monitoring purposes [2]. Often times, data collected from sensors is relayed back to a central processing unit where it is analyzed for trends. In most monitoring applications, the data is collected on a near real-time basis. Early adapters of this wireless technology in industry have been combating several design and performance challenges for a reliably operating system. First and foremost is the issue of data reliability which is intricately linked to the reliability of the communication channel. Interference from other RF sources, such as IEEE 802.11b/g devices or microwave ovens, and multipath effects can severely degrade the performance of the wireless monitoring system. A careful study of all these effects is essential [4]. Another very important issue that needs to be addressed in designing these systems is the power-limited nature of the wireless devices, which is the focus of this paper. In most industrial applications a battery lifespan in the order of several years is required for feasible commercial operation [1]. This requirement imposes severe restrictions on the duration of time the wireless device can be on/awake and the number of transmissions it can make. This is because the radio frequency (RF) communication consumes a significant portion of the battery power when the wireless unit is awake. Therefore, life of the wireless device can be lengthened by optimizing the duty cycle (or reporting frequency) of the unit as well as by transmitting data only when it is necessary.

In this paper, we introduce two conceptual models for wireless sensing with power-limited sensors. The focus is on wireless systems where the sensor can only make a *limited* number of transmissions [5, 6, 7]. The models we consider here are idealized for ease of presentation and mathematical tractability. However, the basic thinking behind these models can easily be adopted to some real-world applications. When doing so, one needs to consider several other requirements imposed on the system, such as communication requirements to keep connectivity and time synchronization, which we ignore in this paper.

In both conceptual models considered, we start with a mathematical description of the process that is under observation. In most applications a model for the process is available, or can be developed from historic data using some regression analysis. In this paper, the process model is assumed to be discrete-time and Markovian [8]. The limited battery power of the wireless device is modeled by imposing a hard constraint on either the number of available transmissions it can make, or on the number of cycles it can stay awake [5, 6, 7]. We think of this hard constraint as a measurement budget, and determine as to how to best spend this budget by scheduling the measurements over a decision horizon.

The rest of the paper is organized as follows. In Section 2, we introduce the problem of optimal scheduling of a finite measurement budget over an observation horizon. Section 3 discusses the optimal estimation problem where the number of transmissions the wireless sensor can make is limited to a number M, which less than the observation horizon, $N > M$. The paper ends with the concluding remarks of Section 4.

2 Optimal Measurement Scheduling with Limited Measurements

In this section, we introduce the problem of estimating a process with limited measurement resources. Under different performance criteria, we show how to best spend a finite measurement budget by scheduling the measurement times over a time horizon.

2.1 Problem Definition

Let $\{X_n, n \geq 0\}$ be a Markov process, where $X_0 = x_0$ is known *a priori*. We would like to measure X_n over a measurement horizon of length N, i.e., $1 \leq n \leq N$, but measurements are expensive. We are given a measurement budget which allows us to make $M < N$ observations of the process. We assume that there is no measurement noise, i.e. when we decide to measure the process we can do it with infinite precision[3].

Let $\{\hat{X}_n, n \geq 0\}$ be the sequence of estimates of the process $\{X_n\}$. Since X_0 is known *a priori*, we have $\hat{X}_0 = x_0$, and $\hat{X}_n = X_n$ for $n \in \mathbf{M}$, where $\mathbf{M} \subset [1, N]$ denotes the set of times a measurement is made. The estimates at other times $n \notin \mathbf{M}$ are determined through an optimization process whereby some estimation error criteria is minimized.

[3] Results of this paper can be extended to the case when there is measurement noise.

In this section, we will consider two types of estimation criteria. The first one is the standard mean-square error criterion where the performance index is the average cumulative estimation error over the decision horizon $[1, N]$:

$$\sum_{n=1}^{N} E\{(X_n - \hat{X}_n)^2\}$$

where the expectation is taken over the statistics of $\{X_n\}$.

We will also consider a "threshold-error" criterion, which is defined as follows[4] . If $\{X_n\}$ is a discrete-state process taking values on $\mathbf{Z}$, we let T_L be the *stopping time*

$$T_L := \inf\{k \geq 1 : X_k \in \mathbf{L}\}$$

where $\mathbf{L} \subset \mathbf{Z}$[5].

The objective of the threshold-error estimation criterion is to have an accurate estimate of the process as it crosses into the threshold set $\mathbf{L}$. Thus, we would like to pick the measurement instances such that the following probability is maximized:

$$P[X_{T_L} = \hat{X}_{T_L} | X_0 = x_0]$$

The continuous-time counterpart of this error criterion is defined when $\{X_n\}$ is a continuous-state process taking values on $\mathbf{R}$. In this case, we define the stopping time

$$N_\tau := \inf\{k \geq 1 : |X_k| \geq \tau\}$$

and determine the time instances we should observe X_n so that the estimation error

$$E\{(X_{N_\tau} - \hat{X}_{N_\tau})^2 | X_0 = x_0\} \tag{1}$$

is minimized.

Finally, we would like to draw attention to the difference between *open-loop* and *closed-loop* measurement system designs. In an open-loop design, the measurement times are determined *a priori* before any value of the process is observed (except $X_0 = x_0$). Since there is no penalty in waiting to decide what time the next measurement should be taken, in a closed-loop design, we wait until a measurement is made to decide on the next measurement time. The advantage of closed-loop design is that it is more robust to process noise, and it can lead to lower values for the performance metrics. Note that the available information about the process increases only when it is observed. Hence, the decision as to when to observe the process next can be made with maximum information at the end of the current observation period. In the context of wireless sensing, this corresponds to the sensor deciding on when to wake-up next before it goes into the sleep-mode to conserve energy.

More precisely, we assume that the information I_n available at time n to decide on the estimate $\hat{X}_n$ is limited to the observed process values up until

[4] The threshold-error criterion is defined over an infinite-time horizon, i.e., $N \to \infty$.
[5] We assume that the Markov process $\{X_n\}$ is irreducible.

time n. So, if no measurement is made between times n and $m > n$, there is no additional information gained, i.e., $I_m = I_n$. Thus, if we measure the process say at time n_k and again at time n_{k+1}, at time n_k we can decide on the sequence of estimates $\hat{X}_n$ between the times n_k and $n_{k+1}-1$. In the case of threshold-error estimation criterion, an additional information structure may be considered where the event $\{X_k \notin \mathbf{L}\}$[6] at time k is observable (measurable) by the decision maker. In this case, at a measurement instance we cannot decide on the sequence of estimates until the next measurement instance, since as the process evolves in time our information about it increases. The time of the next measurement cannot be determined at the time of the current measurement for the same reason. This type of information structure may be well-suited for certain applications, but in the case of wireless sensing, there is no opportunity for the wireless sensors to make intermediate observations about the process between the measurements. Hence, the former information structure, where the event $\{X_k \notin \mathbf{L}\}$ is *not* observable, is more applicable in this case.

In what follows we describe and provide solutions to three class of measurement design problems representative of the more general problems in their respective classes.

2.2 Measurement Schedule Optimization Problems

Problem I

Consider the Gauss-Markov process defined by

$$X_{n+1} = AX_n + W_n, \; n = 0, 1, \ldots \tag{2}$$

where $X_n, A, W_n \in \mathbf{R}$, $X_0 = x_0$ is known, and $\{W_n\}$ is an i.i.d. Gaussian sequence with zero mean and variance σ_w^2. The first problem we consider is estimating the process $\{X_n\}$ over a decision horizon of length N with $M < N$ measurement opportunities. The objective is to minimize the mean-square error

$$e = \sum_{n=1}^{N} E\{(X_n - \hat{X}_n)^2\} \tag{3}$$

As we will see next, this is one of the problems where due to symmetry, the open-loop measurement design will coincide with the closed-loop one. To see this, we first note that if n is a time of measurement, then the estimation error component

$$e_n = E\{(X_n - \hat{X}_n)^2]\}$$

is zero, as $\hat{X}_n = X_n$ for $n \in \mathbf{M}$.

The optimal estimator that minimizes the estimation error e for those times when no measurement is made is given by the conditional expectation

[6] Or $\{|X_k| < \tau\}$ for a continuous-state process.

$$\hat{X}_n = E\{X_n|I_n\},\ n \notin \mathbf{M} \tag{4}$$

where I_n denotes the information available at time n. Since information is obtained only at measurement times, I_n will include the measured process values X_n up to time n. However, due to the Markov nature of the process, knowing the most recent measurement prior to time n is sufficient in determining the conditional expectation in (4). Therefore, we have

$$\hat{X}_n = A^{n-m_n} x_{m_n},\ n \notin \mathbf{M}$$

where m_n denotes the time of the last measurement prior to time n. With this estimator structure, the estimation-error component e_n becomes

$$e_n = \sum_{k=1}^{n-m_n} A^{2(k-1)} \sigma_w^2,\ n \notin \mathbf{M}$$

Note that e_n is *not* a function of the absolute time n, but only of the difference $n - m_n$, i.e., the time since the last measurement. Also, e_n does not depend on any of the measurements made up until time n. Since $e_n = e_{n-m_n}$ increases with the difference $n - m_n$, to minimize the error we must keep this difference as small as possible for all $n \notin \mathbf{M}$. Thus, minimizing the mean-square error e is equivalent to minimizing

$$\mathcal{M} = \sum_{n \notin \mathbf{M}} n - m_n \tag{5}$$

over the M-element subsets of $[1, N]$, i.e., $\mathbf{M}$.

Now, it can be seen that the minimum value of $\mathcal{M}$ is attained when $\mathbf{M}^*$ is the set with its M elements evenly distributed over the measurement interval $[1, N]$. Note that the solution may not be unique, as there may be more than one way to achieve a uniform distribution of M measurement times over the interval $[1, N]$. For example, when $N = 4$ and $M = 2$, $\mathbf{M}^* = \{1, 3\} = \{2, 3\} = \{2, 4\}$ all achieve the minimum $\mathcal{M}^* = 2$.

In summary, the optimal measurement schedule for the Gauss-Markov process (2) can be determined *offline*[7], and is given by a uniform distribution of the measurement opportunities over the measurement horizon.

Problem II

Let $\{X_n, n \geq 0\}$ be a simple walk[8] on integers defined by

$$X_{n+1} = \begin{cases} X_n + 1, & \text{w.p. } p \\ X_n, & \text{w.p. } 1-p \end{cases} \tag{6}$$

[7] Therefore, the open-loop and closed-loop schedules are identical.
[8] One may also consider the symmetric random walk version of this problem.

where $p \in (0,1)$ is the probability of an up-move, and $X_0 = x_0$ is given.

Let T_L be the stopping time

$$T_L := \inf\{k \geq 1 : X_k = L\}$$

where $L > x_0$ is a given integer.

The objective is to detect the process as it crosses the threshold L. Therefore, we want to maximize the probability

$$\mathcal{P} = P[X_{T_L} = \hat{X}_{T_L} | X_0 = x_0\}$$

If we were given an infinite number of observation opportunities, we could make this probability as close to 1 as possible by continuously observing the process. However, measurements are expensive, and therefore we are only allowed to make M of them. In this paper, we only consider the case when $M = 1$, but the results can be extended to an arbitrary $M > 1$. We also assume that the information available at time k to decide on the estimate $\hat{X}_k$ is limited to the observed process values up until time k, i.e., $\{X_k \neq L\}$ is not measurable at time k.

Let $m \geq 1$ be the time of the measurement. The probability $\mathcal{P}$ can be written as

$$\mathcal{P} = P_{x_0}[X_{T_L} = \hat{X}_{T_L}] = \sum_{k=L-x_0}^{\infty} P_{x_0}[X_k = \hat{X}_k] P_{x_0}[T_L = k]$$

We would like to minimize this expression over $m \geq 1$ and $\{\hat{X}_n, n \geq 1\}$. Note that for a given $m \geq 1$, the estimate of X_n that maximizes $\mathcal{P}$ is its maximum likelihood (ML) estimate. For $n < m$, $X_n - x_0$ is Binomial with (n,p), for $n = m$, $X_n = x_m$, and for $n > m$, $X_n - x_m$ is Binomial with $(n-m,p)$. Since the maximum likelihood estimate of a Binomial random variable with parameters (n,p) is given by $\lfloor (n+1)p \rfloor$, we have

$$\hat{X}_n = \begin{cases} x_m + \lfloor (n-m+1)p \rfloor, & n \geq m \\ x_0 + \lfloor (n+1)p \rfloor, & n < m \end{cases} \tag{7}$$

Now, the probabilities $P_{x_0}[X_k = \hat{X}_k]$ can be calculated as follows: for $k < m$

$$P_{x_0}[X_k = \hat{X}_k] = \binom{k}{\lfloor (k+1)p \rfloor} p^{\lfloor (k+1)p \rfloor} (1-p)^{k - \lfloor (k+1)p \rfloor}$$

for $k = m$

$$P_{x_0}[X_k = \hat{X}_k] = 1$$

and for $k > m$:

$$P_{x_0}[X_k = \hat{X}_k] = \binom{k-m}{\lfloor (k-m+1)p \rfloor} p^{\lfloor (k-m+1)p \rfloor} (1-p)^{k-m-\lfloor (k-m+1)p \rfloor}$$

We next calculate $P[T_L = k]$:

$$\begin{aligned} P_{x_0}[T_L = k] &= P_{x_0}[X_{k-1} = L-1, X_k = L] \\ &= P_{x_0}[X_k = L | X_{k-1} = L-1] P_{x_0}[X_{k-1} = L-1] \\ &= p P_{x_0}[X_{k-1} = L-1] \\ &= \binom{k-1}{L-1-x_0} p^{L-x_0} (1-p)^{k-L+x_0} \end{aligned}$$

Now, m can be found by solving the optimization problem

$$\max_{m \geq 1} \sum_{k=L-x_0}^{\infty} P_{x_0}[X_k = \hat{X}_k] \binom{k-1}{L-1-x_0} p^{L-x_0} (1-p)^{k-L+x_0}$$

The solution to this optimization problem depends on the difference $L - x_0$, and a numerical solution can be obtained using Matlab.

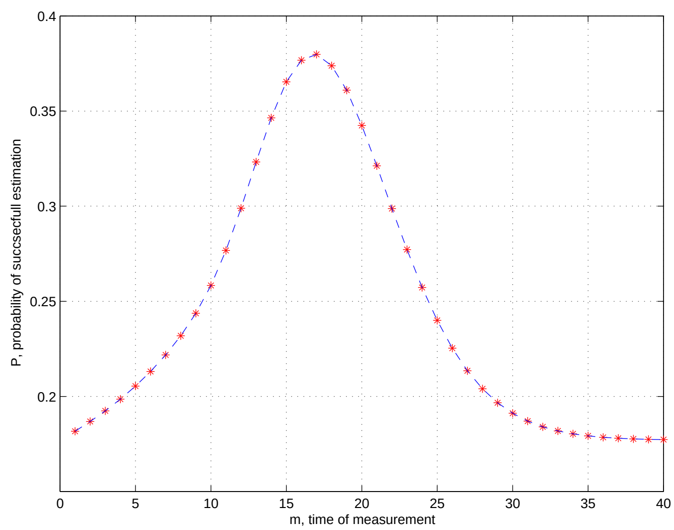

Fig. 1. m vs. $\mathcal{P}$. $p = 0.5, L = 10$

In Figure 1, we plot the successful estimation probability $\mathcal{P}$ at time T_L as a function of the measurement time, $m \geq 1$. The threshold is set at $L = 10$, the walk starts at $x_0 = 0$, and the probability of an up-move is $p = 0.5$. Note that, the probability is maximized when $m^* = 17$. Therefore, if we make a measurement at time $m = 17$, we have approximately 38% chance of being able to catch the process crossing the threshold $L = 10$.

Problem III

Problem III is the continuous-time counterpart of the Problem II. Let X_n be the Gauss-Markov process defined by

$$X_{n+1} = X_n + W_n, \; n = 0, 1, \ldots$$

where $X_0 = x_0$ is known, and $\{W_n\}$ is an i.i.d. Gaussian sequence with zero mean and variance σ_w^2.

Let N_τ be the stopping time

$$N_\tau := \inf\{k \geq 1 : |X_k| \geq \tau\}$$

where $\tau > |x_0|$ is a given threshold. We would like to estimate X_n but again the observations are costly. Say we are allowed to observe the process only once. What time instance should we observe X_n so that the estimation error

$$e(\tau, x_0) = E\{(X_{N_\tau} - \hat{X}_{N_\tau})^2 | X_0 = x_0\} \tag{8}$$

is minimized. In (8), $\hat{X}_n$ denotes the estimate of X_n at time n, and for $n \geq 1$ it is given by[9]

$$\hat{X}_n = \begin{cases} \hat{X}_{n-1}, & m \neq k \\ X_n, & m = k \end{cases} \tag{9}$$

with $\hat{X}_0 = x_0$. In (9), m denotes the time where the observation is made, and we would like to solve the optimization problem:

$$\min_{m \geq 1} E\{(X_{N_\tau} - \hat{X}_{N_\tau})^2 \mid X_0 = x_0\}$$

Conditioning on N_τ, we can equivalently write

$$\min_{m \geq 1} E\{E\{(X_{N_\tau} - \hat{X}_{N_\tau})^2 \mid N_\tau, X_0 = x_0\}\}$$

Now, for a given $m \geq 1$ and $N_\tau \geq 1$, the conditional cost equals

$$E\{(X_{N_\tau} - \hat{X}_{N_\tau})^2 \mid N_\tau, m, X_0 = x_0\} = \begin{cases} (N_\tau - m)\sigma_w^2, & 1 \leq m < N_\tau \\ 0, & m = N_\tau \\ N_\tau \sigma_w^2, & m > N_\tau \end{cases}$$

Hence, the average cost for $m \geq 2$ can be written as

$$\begin{aligned} e_m(\tau, x_0) = & \sum_{k=1}^{m-1} k P[N_\tau = k | X_0 = x_0] \sigma_w^2 \\ & + \sum_{k=m+1}^{\infty} (k - m) P[N_\tau = k | X_0 = x_0] \sigma_w^2 \end{aligned}$$

[9] Since the event $\{|X_k| < \tau\}$ is not measurable at time k.

and for $m = 1$

$$e_1(\tau, x_0) = \sum_{k=m+1}^{\infty} (k-m) P[N_\tau = k | X_0 = x_0] \sigma_w^2$$

Note that for $m \geq 2$, we have

$$\begin{aligned} e_m(\tau, x_0) &= E\{N_\tau | X_0 = x_0\} \sigma_w^2 - m \sum_{k=m}^{\infty} P[N_\tau = k | X_0 = x_0] \sigma_w^2 \\ &= E\{N_\tau | X_0 = x_0\} \sigma_w^2 - m \left(1 - \sum_{k=1}^{m-1} P[N_\tau = k | X_0 = x_0]\right) \sigma_w^2 \end{aligned}$$

and for $m = 1$

$$e_1(\tau, x_0) = E\{N_\tau | X_0 = x_0\} \sigma_w^2 - \sigma_w^2$$

Since $\{X_k\}$ is a Markov chain, we write $P[N_\tau = k | X_0 = x_0]$ as

$$\begin{aligned} P[N_\tau = k | X_0 = x_0] &= P_{x_0}[|X_k| \geq \tau, |X_{[1,k]}| < \tau] \\ &= P_{x_0}[|X_k| \geq \tau, |X_1| < \tau, \ldots, |X_{k-1}| < \tau] \\ &= P_{x_0}[|X_k| \geq \tau | |X_{k-1}| < \tau] \\ &\quad \cdots P_{x_0}[|X_2| < \tau | |X_1| < \tau] P_{x_0}[|X_1| < \tau] \end{aligned}$$

Let $p_k(\tau, x_0)$ denote the conditional probability

$$p_k(\tau, x_0) = P[|X + W| < \tau | |X| < \tau]$$

where $W \sim N(0, \sigma_w^2)$, and $X \sim N(x_0, k\sigma_w^2),\ k \geq 1$, and (X, W) are independent. By definition

$$p_k(\tau, x_0) = \frac{P[|X + W| < \tau, |X| < \tau]}{P[|X| < \tau]}$$

Now,

$$P[|X| < \tau] = \Phi\left(\frac{\tau - x_0}{\sqrt{k}\sigma_w}\right) - \Phi\left(\frac{-\tau - x_0}{\sqrt{k}\sigma_w}\right)$$

where $\Phi(\cdot)$ is the CDF of the standard Gaussian random variable. Also, note that

$$\begin{aligned} &P[|X + W| < \tau, |X| < \tau] \\ &= \int_{-\tau}^{\tau} \frac{1}{\sqrt{2\pi k \sigma_w^2}} e^{-\frac{(x - x_0)^2}{2k\sigma_w^2}} \left[\Phi\left(\frac{\tau - x}{\sigma_w}\right) - \Phi\left(\frac{-\tau - x}{\sigma_w}\right)\right] dx \end{aligned}$$

Thus,

$$p_k(\tau, x_0) = \frac{\int_{-\tau}^{\tau} \frac{1}{\sqrt{2\pi k \sigma_w^2}} e^{-\frac{(x - x_0)^2}{2k\sigma_w^2}} \left[\Phi\left(\frac{\tau - x}{\sigma_w}\right) - \Phi\left(\frac{-\tau - x}{\sigma_w}\right)\right] dx}{\Phi\left(\frac{\tau - x_0}{\sqrt{k}\sigma_w}\right) - \Phi\left(\frac{-\tau - x_0}{\sqrt{k}\sigma_w}\right)}$$

Let $p_0(\tau, x_0)$ denote

$$p_0(\tau, x_0) = P[|X_1| < \tau | X_0 = x_0] = \Phi\left(\frac{\tau - x_0}{\sigma_w}\right) - \Phi\left(\frac{-\tau - x_0}{\sigma_w}\right)$$

Using $p_k(\tau, x_0)$'s, we write the probability distribution of N_τ as

$$P[N_\tau = k | X_0 = x_0] = \begin{cases} 1 - p_0(\tau, x_0), & k = 1 \\ (1 - p_{k-1}(\tau, x_0)) \prod_{n=0}^{k-2} p_n(\tau, x_0), & k \geq 2 \end{cases}$$

Substituting the expression for $P[N_\tau = k | X_0 = x_0]$ into the error expression $e_m(\tau, x_0)$ for $m \geq 3$ yields

$$\begin{aligned} e_m(\tau, x_0) = {} & E\{N_\tau | X_0 = x_0\}\sigma_w^2 \\ & -m\left(1 - \sum_{k=1}^{m-1}(1 - p_{k-1}(\tau, x_0)) \prod_{n=0}^{k-2} p_n(\tau, x_0)\right)\sigma_w^2 \end{aligned}$$

and for $m = 1, 2$, we have

$$\begin{aligned} e_1(\tau, x_0) &= E\{N_\tau | X_0 = x_0\}\sigma_w^2 - \sigma_w^2 \\ e_2(\tau, x_0) &= E\{N_\tau | X_0 = x_0\}\sigma_w^2 - 2p_0(\tau, x_0)\sigma_w^2 \end{aligned}$$

We next look at the normalized difference

$$\delta_m(\tau, x_0) := \frac{e_{m+1}(\tau, x_0) - e_m(\tau, x_0)}{\sigma_w^2}$$

For $m \geq 2$, calculating this difference yields

$$\delta_m(\tau, x_0) = mP[N_\tau = m | X_0 = x_0] + \sum_{k=1}^{m} P[N_\tau = k | X_0 = x_0] - 1$$

and for $m = 1$, we have

$$\delta_1(\tau) = 1 - 2p_0(\tau, x_0)$$

Note that, by telescoping the last term, $\delta_m(\tau, x_0), m \geq 2$ can be written as

$$\begin{aligned} \delta_m(\tau, x_0) &= m(1 - p_{m-1}(\tau, x_0)) \prod_{n=0}^{m-2} p_n(\tau, x_0) - 1 \\ &\quad + \sum_{k=1}^{m}(1 - p_{k-1}(\tau, x_0)) \prod_{n=0}^{k-2} p_n(\tau, x_0) \\ &= m(1 - p_{m-1}(\tau, x_0)) \prod_{n=0}^{m-2} p_n(\tau, x_0) - 1 + 1 - \prod_{n=0}^{m-1} p_n(\tau, x_0) \\ &= m \prod_{n=0}^{m-2} p_n(\tau, x_0) - (m+1) \prod_{n=0}^{m-1} p_n(\tau, x_0) \end{aligned}$$

The error sequence $\{e_m(\tau, x_0)\}$ for $m \geq 1$ is decreasing if and only if

$$\delta_m(\tau, x_0) < 0$$

Therefore, the estimation error is minimum for $m^*(\tau, x_0)$ such that

$$m^*(\tau, x_0) = \inf\{m \geq 1 : \delta_m(\tau, x_0) > 0\}$$

The comparison $\delta_m(\tau, x_0) > 0$ is equivalent to, for $m = 1$

$$\delta_1(\tau, x_0) > 0 \Leftrightarrow 1 - 2p_0(\tau, x_0) > 0 \Leftrightarrow p_0(\tau, x_0) < \frac{1}{2}$$

and for $m \geq 2$

$$\begin{aligned} \delta_m(\tau, x_0) > 0 &\Leftrightarrow m \prod_{n=0}^{m-2} p_n(\tau, x_0) - (m+1) \prod_{n=0}^{m-1} p_n(\tau, x_0) > 0 \\ &\Leftrightarrow p_{m-1}(\tau, x_0) < \frac{m}{m+1} \end{aligned}$$

Hence, to minimize the estimation error we pick $m^*(\tau, x_0)$ such that

$$m^*(\tau, x_0) = \inf\left\{m \geq 1 : p_{m-1}(\tau, x_0) < \frac{m}{m+1}\right\}$$

Note that a solution always exists, since $p_m(\tau, x_0) \downarrow$ as $m \uparrow$, and $\frac{m}{m+1} \to 1$, as $m \to \infty$. This feature of the solution is illustrated in Figure 2, where both $p_{m-1}(\tau, x_0)$ (red-square) and the function $\frac{m}{m+1}$ (blue-diamond) are plotted against m. In Figure 2, the threshold is set as $\tau = 3$, $x_0 = 0$, and $\sigma_w^2 = 1$. For these parameters, the optimal measurement time is given by $m^* = 9$.

3 Optimal Estimation with Limited Measurements

In this section, we turn our attention into a sequential estimation problem with two decision makers who work as members of a team [6]. One of the decision makers is the wireless sensor and it makes sequential measurements about the state of an underlying stochastic process for a fixed period of time. Note that this is different than the setup considered in Section 2 where the wireless sensor schedules its measurements across time each time before it goes to sleep. The sensor (or observer) upon measuring the process makes a decision as to whether to transmit some information about the process to the estimator. The estimator sequentially estimates the state of the process. The objective is to minimize a performance criterion with the constraint that the sensor may only transmit a limited number of measurements.

More specifically, we consider estimating a stochastic process over a decision horizon of length N using only $M \leq N$ measurements. Both the measurement and estimation of the process is carried out sequentially by two different

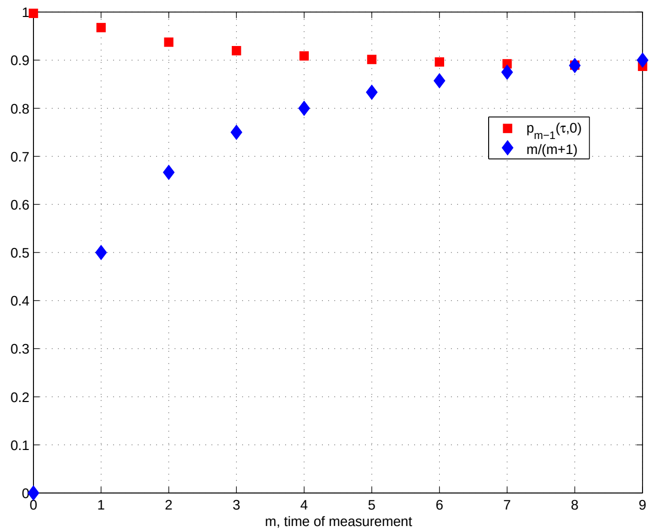

Fig. 2. m vs. $p_{m-1}(\tau, x_0)$ (red-square) and $\frac{m}{m+1}$ (blue-diamond). $x_0 = 0, \tau = 3$

decision makers called the *observer* and the *estimator*[10], respectively. Over the decision horizon of length N, the observer agent has exactly M opportunities to disclose some information about the process to the estimator. These information disclosures, or transmissions, are assumed to be error and noise free, and the problem is to jointly determine the best observation and estimation policies that minimize the average estimation error between the process and its estimate.

3.1 Problem Statement

Problem Definition

The problem of optimal estimation with limited measurements can be treated in the more general framework of a communication system with limited channel uses. For this purpose, consider the generic communication system whose block diagram is given in Figure 3 [9]. The source outputs some data b_k for $0 \leq k \leq N-1$, that needs to be communicated to the user over a channel. The data b_k are generated according to some *a priori* known stochastic process, $\{b_k\}$, which may be i.i.d., or correlated as in a Markov process. An encoder (or an observer) and a decoder (or an estimator) is placed after the source output and the channel output, respectively, to communicate the data to the user efficiently. In the most general case, the encoder/observer may have access to a noise-corrupted version of the source output:

[10] As we show next, in a communication-theoretic setting we may call them an *encoder* and a *decoder*, respectively.

$$z_k = b_k + v_k, \ 0 \le k \le N-1$$

where $\{v_k\}$ is an independent[11] noise process.

The main constraint is that the encoder/observer can access the channel only a *limited*, $M \le N$, number of times. The goal is to design an observer-estimator pair[12], $(\mathcal{O}, \mathcal{E})$, that will "causally" (or sequentially) observe/encode the data measurements, z_k, and estimate/decode the channel output, y_k, so as to minimize the *average distortion* or *error* between the observed data, b_k, and estimated data, $\hat{b}_k$.

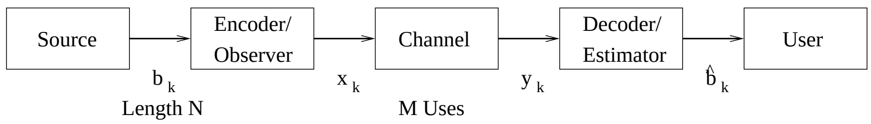

Fig. 3. Communication with limited channel use.

The channel is assumed to be memoryless, and is completely characterized by the conditional probability distribution $P_c(y|x)$ on $y \in \mathcal{Y}$ for each $x \in \mathcal{X}$, where $\mathcal{X}$ and $\mathcal{Y}$ are the set of allowable channel inputs, and the set of possible channel outputs, respectively.

The average distortion $D_{(M,N)}$ depends on the distortion measure and may vary depending on the underlying application. Some examples are the average mean-square error

$$D_{(M,N)} = E\left\{ \frac{1}{N} \sum_{k=0}^{N-1} (b_k - \hat{b}_k)^2 \right\} \tag{10}$$

or the Hamming (probability of error) distortion measure

$$D_{(M,N)} = E\left\{ \frac{1}{N} \sum_{k=0}^{N-1} \mathcal{I}_{b_k \neq \hat{b}_k} \right\} \tag{11}$$

where $\mathcal{I}_S$ denotes the indicator function of the set S.

From a communication-theoretic standpoint, with the channel, source, and the distortion measure defined, we can formally state our main problem: Given a source and a memoryless channel, for a given decision-horizon N, and number of channel uses M, what is the minimum attainable value of the average distortion $D_{(M,N)}$? This minimization is carried out over the choice of possible encoder-decoder (observer-estimator) pairs which are *causal*.

In this paper, we present a solution to this problem when the source process is i.i.d. with a continuous or discrete probability density function, and the

[11] Independent across time and from the source output process b_k.
[12] Or depending on the application, an encoder-decoder pair $(\mathcal{E}, \mathcal{D})$.

encoder/observer has access to the noiseless or a noisy version of the source output. We assume that the channel is noiseless, and hence, it is completely characterized by the probability distribution $P_c(y|x) = \delta(y - x)$. We also present the solution to the case when the source process is Gauss-Markov.

Note that, in wireless sensing applications the desired length of time the wireless device will be in operation can be related to the decision horizon N in some appropriate time unit, and the size of the battery installed in the sensor can be related to the possible number of transmissions or channel uses M (see Figure 4).

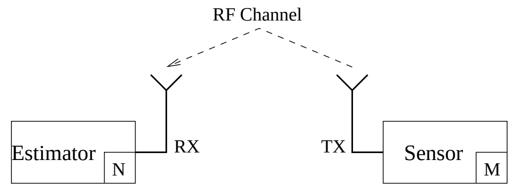

Fig. 4. Optimal transmission scheduling with limited channel access.

Hence, given an underlying performance criterion $D_{(M,N)}$, the problem is to design the best transmission schedule, and estimation policies for the wireless device and the remote monitoring station, respectively.

3.2 Estimating an i.i.d. Random Sequence with Limited Measurements

Problem Definition

Consider the special case of the general problem defined in Section 3.1, where the source outputs a zero-mean[13] i.i.d. random sequence $b_k,\ 0 \leq k \leq N-1$. Let $\mathcal{B}$ denote the range of the random variable b_k. We assume that b_k's have a finite second moment, $\sigma_b^2 < \infty$, but their probability distribution remains unspecified for now. At time k, the encoder/observer makes a sequential measurement of b_k, and determines whether to access the channel for transmission, which it can only do a limited, $M \leq N$, number of times. The channel is noiseless and thus has a capacity to transmit the source output error-free when it is used to transmit. Note that, even when it decides not to use the channel for transmission, the observer/encoder may still convey a 1-bit information to the estimator/decoder. In view of this, the channel input x_k belongs to the set $\mathcal{X} := \mathcal{B} \cup \{\text{NT}\}$, where NT stands for "no transmission."

More precisely, we let s_k denote the number of channel uses (or transmissions) left at time k. Now if $s_k \geq 1$, we have $y_k = x_k$ for $x_k \in \mathcal{B} \cup \{\text{NT}\}$. If $s_k = 0$, on the other hand, the channel is useless, since we have exhausted the

[13] This is not restrictive, as the known mean can be subtracted out by the estimator.

allocated number of channel uses. Note that, when the channel is noiseless, both the encoder and the decoder can keep track of s_k by initializing $s_0 = M$ and decrementing it by 1 every time a transmission decision is taken.

We want to design an estimator/decoder

$$\hat{b}_k = \hat{\mu}_k(I_k^d) \text{ for } 0 \leq k \leq N-1$$

based on the available information I_k^d at time k. Clearly, the information available to the estimator is controlled by the observer. The average distortion between the observed and estimated processes can be taken to be the average mean square error as given by (10), or the probability of error distortion measure which is given by (11).

The information I_k^d available to the estimator at time k is a result of an outcome of decisions taken by the observer up until time k. Let the observer's decision at time k be

$$x_k = \mu_k(I_k^e)$$

where I_k^e is the information available to the observer at time k. Assuming perfect recall, we have

$$\begin{aligned} I_0^e &= \{(s_0, t_0); b_0\} \\ I_k^e &= \{(s_k, t_k); b_0^k; x_0^{k-1}\},\ 1 \leq k \leq N-1 \end{aligned}$$

where t_k denotes the number of time, or decision slots left at time k. We have

$$t_{k+1} = t_k - 1,\ 0 \leq k \leq N-2$$

with $t_0 = N$.

The range of $\mu_k(\cdot)$ is the space $\mathcal{X} = \mathcal{B} \cup \{\text{NT}\}$. Let σ_k denote the decision whether the observer has decided to transmit or not. Assume $s_k \geq 1$, and let $\sigma_k = 1$ if a transmission takes place; i.e., $x_k \in \mathcal{B}$, and $\sigma_k = 0$ if no transmission takes place. We have

$$s_{k+1} = s_k - \sigma_k,\ 0 \leq k \leq N-2$$

with $s_0 = M$.

The observer's decision at time k is a function of its k past measurements, and $k-1$ past decisions, i.e.,

$$\mu_k(I_k^e) : \mathcal{B}^k \times \mathcal{X}^{k-1} \to \mathcal{X},\ 0 \leq k \leq N-1$$

Now, the information I_k^d available to the estimator at time k can be written as

$$I_k^d = \{(s_k, t_k); y_0^k\},\ 0 \leq k \leq N-1$$

By definition, the channel output y_k satisfies $y_k = x_k$ if $s_k \geq 1$, and $y_k \in \emptyset$ (i.e., no information) if $s_k = 0$.

Consider the class of observer-estimator (encoder-decoder) policies consisting of a sequence of functions

$$\Pi = \{\mu_0, \hat{\mu}_0, \ldots, \mu_{N-1}, \hat{\mu}_{N-1}\}$$

where each function μ_k maps I_k^e into $\mathcal{X}$, and $\hat{\mu}_k$ maps I_k^d into $\mathcal{B}$[14], with the additional restriction that μ_k can map to $\mathcal{B}$ at most M times. Such policies are called *admissible*.

We want to find an admissible policy $\pi^* \in \Pi$ that minimizes the average N-stage distortion, or estimation error:

$$e^{\pi}_{(M,N)} = E\left\{\sum_{k=0}^{N-1} (b_k - \hat{\mu}_k(I_k^d))^2\right\} \tag{12}$$

or for source processes, b_k, with discrete probability densities:

$$e^{\pi}_{(M,N)} = E\left\{\sum_{k=0}^{N-1} \mathcal{I}_{b_k \neq \hat{\mu}_k(I_k^d)}\right\} \tag{13}$$

That is

$$e^{*}_{(M,N)} = \min_{\pi \in \Pi} e^{\pi}_{(M,N)} \tag{14}$$

Note that, we omitted the factor of $\frac{1}{N}$ from the average error expressions for convenience.

If $M \geq N$, this problem has the trivial solution where the observer writes the source output b_k directly into the channel at each time k (i.e., $\mu_k^*(b_k) = b_k$), and since the channel is noiseless, the estimator can use an identity mapping (i.e., $\hat{\mu}_k^*(I_k^d) = b_k$), resulting in zero distortion. Therefore, we only consider the case when $M < N$.

Before closing our account on this section, we would like to note the non-classical nature of the information in this problem. Clearly, the observer's action affects the information available to the estimator, and there is no way in which the estimator can infer the information available to the observer. Also note the order of actions between the decision makers in the problem. At time k, first the random variable b_k becomes available, then the observer acts by transmitting some data or not, and finally, the estimator acts by estimating the state with $\hat{\mu}_k$, the cost is incurred, and we move to the next time $k+1$.

Structure of the Solution

We first consider the problem of finding the optimal estimator $\hat{\mu}_k^*$ at time k. Note that the estimator $\hat{\mu}_k$ appears only in a single term in the error

[14] Note that we do not distinguish between the source and user sets.

expressions (12)-(13). Thus, for the mean-square error criterion, the optimal estimator is simply the solution of the quadratic minimization problem

$$\min_{\hat{\mu}_k(I_k^d)} E\left\{(b_k - \hat{\mu}_k(I_k^d))^2 | I_k^d\right\} \tag{15}$$

which is given by the conditional expectation of b_k given the available information at time k:

$$\hat{\mu}_k^*(I_k^d) = E\{b_k | I_k^d\} = E\{b_k | (s_k, t_k); y_0^k\} \tag{16}$$

Similarly, for the probability of error distortion criterion, the optimal estimator is the solution of the minimization problem

$$\min_{\hat{\mu}_k(I_k^d)} E\left\{\mathcal{I}_{b_k \neq \hat{\mu}_k(I_k^d)} | I_k^d\right\}$$

If at time k the channel can still be used ($s_k \geq 1$), the solution to this problem is given by the maximum *a posteriori* probability (MAP) estimate of the random variable b_k given the available information at time k:

$$\hat{\mu}_k^*(I_k^d) = \arg \max_{m_i \in \mathcal{B}_k(I_k^d)} \delta(y_k - i) p_i = \arg \max_{m_i \in \mathcal{B}_k((s_k, t_k); y_0^k)} p_i \tag{17}$$

where $\mathcal{B}_k(I_k^d) \subset \mathcal{B}$ is some subset of the range of the random variable b_k, which we assume is countable. Let m_i denote the values the random variable b_k takes. Then, p_i's denote the probability mass function of the random variable b_k, i.e., $p_i = P[b_k = m_i]$.

Note that, for the probability of error distortion criterion, if the channel is useless at time k (i.e., $s_k = 0$), the best estimate of b_k is simply given by

$$\hat{\mu}_k^*(I_k^d) = \arg \max_{m_i \in \mathcal{B}} p_i \tag{18}$$

since the past channel outputs, y_0^{k-1}, are independent of b_k.

Similarly, for the mean-square error criterion, the channel output y_k has no information on b_k if $s_k = 0$. Thus, in this case, the conditional expectation in (16) equals

$$\hat{\mu}_k^*(I_k^d) = E\{b_k | (0, t_k); y_0^{k-1}, y_k\} = E\{b_k\} = 0 \tag{19}$$

since again the past channel outputs, y_0^{k-1}, are generated by the σ-algebra of random variables b_0^{k-1}, and hence are independent from b_k.

If $s_k \geq 1$, the channel output $y_k = x_k$, but since $y_0^{k-1} = x_0^{k-1}$ is the outcome of a Borel-measurable function defined on the σ-algebra generated by b_0^{k-1}, the conditional expectation in (16) is equivalent to

$$\hat{\mu}_k^*(I_k^d) = E\{b_k | (s_k, t_k); x_k\} \tag{20}$$

By a similar argument we can write (17) as

$$\hat{\mu}_k^*(I_k^d) = \arg \max_{m_i \in \mathcal{B}_k((s_k,t_k);x_k)} p_i \tag{21}$$

Now, substituting the optimal estimators (20)-(21) back into the estimation error expressions (12)-(13) yields

$$e_{(M,N)}^{\pi} = E\left\{\sum_{k=0}^{N-1} (b_k - E\{b_k|(s_k,t_k);x_k\})^2\right\} \tag{22}$$

and

$$e_{(M,N)}^{\pi} = E\left\{\sum_{k=0}^{N-1} \mathcal{I}_{b_k \neq arg\max_{m_i \in \mathcal{B}_k((s_k,t_k);x_k)} p_i}\right\} \tag{23}$$

which we seek to minimize over the observer/encoder policies $\mu_k(I_k^e), 0 \leq k \leq N-1$. Since $x_k = \mu_k(I_k^e)$, we see that the choice of an observer policy affects the cost only through the information made available to the estimator.

In general, the observer's decision μ_k at time k depends on (s_k,t_k), all past measurements b_0^{k-1}, the present measurement b_k, and its past actions x_0^{k-1}. However, as we show next, there is nothing the observer can gain by having access to its past measurements b_0^{k-1} and its past actions x_0^{k-1} as far as the optimization of the criteria (22)-(23) are concerned. Thus, a *sufficient statistics* for the observer are the current measurement b_k and the remaining number of channel uses (transmission opportunities) and decision instances, i.e. (s_k,t_k).

Proposition 1. *The set $S_k^e = \{(s_k,t_k);b_k\}$ constitutes sufficient statistics $S_k^e(I_k^e)$ for the optimal policy μ_k^* of the observer. In other words,*

$$\mu_k^*(I_k^e) = \bar{\mu}(S_k^e(I_k^e))$$

for some function $\bar{\mu}$.

Proof. Suppose we would like to determine the optimal observer policy $\mu_k^*(I_k^e)$ at time k, where $0 \leq k \leq N-1$ is arbitrary. Due to the sequential nature of the decision problem, any observer policy we decide on at time k will only affect the error e_k incurred after time k, i.e.[15],

$$e_k = E\left\{\sum_{n=k}^{N-1} (b_n - E\{b_n|(s_n,t_n);x_n\})^2\right\}$$

Taking the conditional expectation given the available information I_k^e, under any observer policy $\mu_k(I_k^e)$ we have

$$E\{e_k|(s_k,t_k);b_0^k;x_0^{k-1}\} = E\{e_k|(s_k,t_k);b_k\} \tag{24}$$

because b_k^{N-1} is independent of b_0^{k-1}, and x_0^{k-1} is the outcome of a Borel-measurable function defined on the σ-algebra generated by b_0^{k-1}. Hence, at time k, the knowledge of b_0^{k-1} and x_0^{k-1} is redundant.

[15] Here, we give the proof only for the error criterion (22). An identical proof can be constructed for the probability of error distortion criterion (23).

A consequence of Proposition 1 is that the observer's decision to use the channel to transmit a source measurement or not is based purely on the current observation b_k and its past actions only through (s_k, t_k).

Since μ_k depends explicitly only on the current source output b_k, the search for an optimal observer policy can be narrowed down to the class of policies of the form[16]

$$\mu_k(I_k^e) = \bar{\mu}((s_k, t_k); b_k) = \begin{cases} b_k & \text{if } b_k \in \mathcal{T}_{(s_k,t_k)} \\ \text{NT} & \text{if } b_k \in \mathcal{T}^c_{(s_k,t_k)} \end{cases} \tag{25}$$

where $\mathcal{T}_{(s_k,t_k)}$ is a measurable set on $\mathcal{B}$ and is a function of (s_k, t_k). The complement of the set $\mathcal{T}_{(s_k,t_k)}$ is taken with respect to $\mathcal{B}$, i.e., $\mathcal{T}^c_{(s_k,t_k)} = \mathcal{B} \backslash \mathcal{T}_{(s_k,t_k)}$. When probability of error distortion criterion is used, Proposition 1 implies that $\mathcal{B}_k((s_k, t_k); \text{NT}) = \mathcal{T}^c_{(s_k,t_k)}$, and $\mathcal{B}_k((s_k, t_k); m_i) = m_i$.

Note that the optimal estimators (20) and (21) have access to (s_k, t_k) as well. Thus, even when the observer chooses not to transmit b_k, it can still pass a 1-bit information about b_k to the estimator provided that $s_k \geq 1$. If k is such that all M transmissions are concluded prior to time k (i.e., $s_k = 0$), the estimators are given by (18)-(19), irrespective of b_k.

Now, observe that the optimization over the observer policies is equivalent to optimization over the sets $\mathcal{T}_{(s_k,t_k)}$ for all k such that

$$\max\{0, M - k\} \leq s_k \leq \min\{t_k, M\}$$

and $t_k = N - k$. The nonnegativity of s_k is a result of the limited channel use constraint. Note that if $s_{k_0} = 0$ for some k_0, then $s_k = 0$ for all k such that $k_0 \leq k \leq N - 1$. At the other extreme, we must have $s_k \leq N - k$, since if $s_k = N - k$, this means there are as many channel uses left as there are decision instances, and the optimal observer and estimator policies in this case are obvious.

The Solution with the Mean-Square Error Criterion

Let $(s_k, t_k) = (s, t)$, and $e^*_{(s,t)}$ denote the optimal value of the estimation error (or distortion) (22) when the decision horizon is of length t, and the observer is limited to s channel uses, where $s \leq t$. We know that at time k, the optimal observation policy will be of the form (25).

Now, at time $k + 1$, depending on the realization of the random variable b_k, the remaining $(t-1)$-stage estimation error is either $e^*_{(s-1,t-1)}$, or $e^*_{(s,t-1)}$. Thus, inductively by the DP equation [10], we can write[17]

[16] As long as k is such that all M measurements are not exhausted, i.e., $s_k \geq 1$.

[17] Assuming that the random variables $\{b_k\}$ are continuous with a well-defined probability density function (pdf) $f(b)$.

$$e^*_{(s,t)} = \min_{\mathcal{T}_{(s,t)}} \left\{ e^*_{(s-1,t-1)} \int_{b\in\mathcal{T}_{(s,t)}} f(b)db + e^*_{(s,t-1)} \int_{b\in\mathcal{T}^c_{(s,t)}} f(b)db + \int_{b\in\mathcal{T}^c_{(s,t)}} \left[b - E\{b|b\in\mathcal{T}^c_{(s,t)}\}\right]^2 f(b)db \right\}$$

where $f(b)$ is the pdf of the random variables b_k. If b_k's are discrete random variables with a probability mass function (pmf), one has to replace the integrals in the above expression with sums. Expanding out the expectation yields

$$e^*_{(s,t)} = \min_{\mathcal{T}_{(s,t)}} \left\{ e^*_{(s-1,t-1)} \int_{b\in\mathcal{T}_{(s,t)}} f(b)db + e^*_{(s,t-1)} \int_{b\in\mathcal{T}^c_{(s,t)}} f(b)db + \int_{b\in\mathcal{T}^c_{(s,t)}} \left[b - \frac{\int_{b\in\mathcal{T}^c_{(s,t)}} bf(b)db}{\int_{b\in\mathcal{T}^c_{(s,t)}} f(b)db}\right]^2 f(b)db \right\} \tag{26}$$

To solve for $e^*_{(s,t)}$, we first note the boundary conditions $e^*_{(t,t)} = 0$, and $e^*_{(0,t)} = t\sigma_b^2$, $\forall t \geq 0$, where σ_b^2 is the variance of b_k. The term $e^*_{(s,t)}$ remains undefined for $s > t$. The optimal sets satisfy the boundary conditions $\mathcal{T}^*_{(t,t)} = \mathcal{B}$, and $\mathcal{T}^*_{(0,t)} = \emptyset$, $\forall t \geq 0$. The recursion of (26) needs to be solved offline and the optimal sets $\mathcal{T}^*_{(s,t)}$ must be tabulated starting with smaller values of (s,t)[18]. The solution to the original problem can then be determined as follows:

Initialize $s_0 = M$, $t_0 = N$. For each k in $0 \leq k \leq N-1$ do the following:

1. Look up the optimal set $\mathcal{T}^*_{(s_k,t_k)}$ from the table that was determined offline.
2. Observe b_k, and apply the observation policy

$$\bar{\mu}^*((s_k,t_k);b_k) = \begin{cases} b_k & \text{if } b_k \in \mathcal{T}^*_{(s_k,t_k)} \\ \text{NT} & \text{if } b_k \in \mathcal{T}^{*c}_{(s_k,t_k)} \end{cases}$$

3. Apply the estimation policy

$$\hat{\mu}^*_k(\mathcal{T}^*_{(s_k,t_k)}) = E\{b_k|b_k \in \mathcal{T}^{*c}_{(s_k,t_k)}\} = \frac{\int_{b\in\mathcal{T}^{*c}_{(s_k,t_k)}} bf(b)db}{\int_{b\in\mathcal{T}^{*c}_{(s_k,t_k)}} f(b)db}$$

4. Update

$$s_{k+1} = s_k - \sigma_k,\ t_{k+1} = t_k - 1$$

In tabulating $\mathcal{T}^*_{(s,t)}$ one should start with solving for $\mathcal{T}^*_{(1,2)}$, and the corresponding estimation error $e^*_{(1,2)}$. To determine the optimal set at (s,t), we need to know the optimal costs at $(s,t-1)$, and $(s-1,t-1)$. Hence, we can propagate our calculations as shown in Figure 5 starting with $(s,t) = (1,2)$.

[18] Note that $(1,2)$ is the smallest possible nontrivial value.

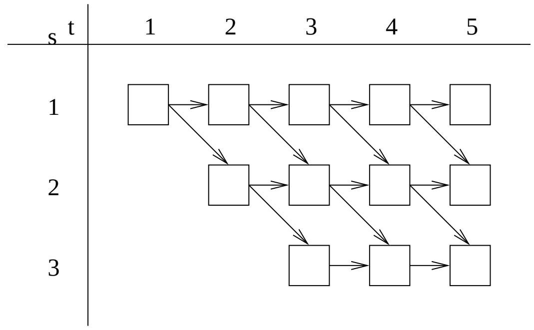

Fig. 5. Recursive calculation of $e^*_{(s,t)}$.

Now, we come back to the problem of minimizing (26) over $\mathcal{T}_{(s,t)}$. Expanding out the expression inside the minimization we get

$$e^*_{(s,t)} = e^*_{(s-1,t-1)} + \min_{\mathcal{T}^c_{(s,t)}} \left\{ -\left(e^*_{(s-1,t-1)} - e^*_{(s,t-1)}\right) \int_{b\in\mathcal{T}^c_{(s,t)}} f(b)db + \int_{b\in\mathcal{T}^c_{(s,t)}} b^2 f(b)db - \frac{\left[\int_{b\in\mathcal{T}^c_{(s,t)}} bf(b)db\right]^2}{\int_{b\in\mathcal{T}^c_{(s,t)}} f(b)db} \right\} \quad (27)$$

where we used the fact that $\int_{b\in\mathcal{T}_{(s,t)}} f(b)db = 1 - \int_{b\in\mathcal{T}^c_{(s,t)}} f(b)db$.

This an optimization problem over measurable sets $\mathcal{T}^c_{(s,t)}$ on the real line, and since these sets are not countable, there is no known method for carrying out this minimization in a systematic manner. Therefore, we restrict our search to the sets that are in the form of simple symmetric intervals, i.e., $\mathcal{T}^c_{(s,t)} = [-\beta_{(s,t)}, \beta_{(s,t)}]$, where $0 \le \beta_{(s,t)} \le \infty$.

Now, because of symmetry, the last term on the right-hand side of (27) disappears from the minimization. Differentiating the remaining terms inside the curly brackets, we obtain the first-order necessary condition:

$$-\left(e^*_{(s-1,t-1)} - e^*_{(s,t-1)}\right) f(\beta_{(s,t)}) + \beta^2_{(s,t)} f(\beta_{(s,t)}) = 0 \quad (28)$$

From which the critical point $\beta^*_{(s,t)}$ can be determined as[19]

$$\beta^*_{(s,t)} = \sqrt{e^*_{(s-1,t-1)} - e^*_{(s,t-1)}} \quad (29)$$

Note that, we always have $e^*_{(s,t-1)} \le e^*_{(s-1,t-1)}$, since for the same decision horizon, $t-1$, the minimum average distortion achieved by s channel uses, is always less than that achieved by $s-1$ channel uses. So, $\beta^*_{(s,t)}$ always exists.

[19] The other critical point, namely $\beta_{(s,t)} = +\infty$, yields a larger cost.

From the first-order condition, we observe that the objective function is strictly decreasing on the interval $[0, \beta^*_{(s,t)})$, and it is strictly increasing on the interval $(\beta^*_{(s,t)}, \infty)$. Thus, $\beta^*_{(s,t)}$ must be a strict global minimizer. Thus, in the class of symmetric intervals, the best set $\mathcal{T}^c_{(s,t)}$ is given by the interval

$$\mathcal{T}^{*c}_{(s,t)} = [-\sqrt{e^*_{(s-1,t-1)} - e^*_{(s,t-1)}}, \sqrt{e^*_{(s-1,t-1)} - e^*_{(s,t-1)}}] \tag{30}$$

The Solution with the Probability of Error Criterion

As in Section 3.2, let $(s_k, t_k) = (s, t)$, and let $e^*_{(s,t)}$ denote the optimal value of the estimation error (or distortion) (23) when the decision horizon is of length t, and the observer is limited to s channel uses, where $s \leq t$. We know that at time k, the optimal observation (transmission) policy will be of the form (25).

Now, at time $k+1$, depending on the realization of the random variable b_k, the remaining $(t-1)$-stage estimation error is either $e^*_{(s-1,t-1)}$, or $e^*_{(s,t-1)}$. Thus, assuming that $s \geq 1$, inductively by the DP equation, we can write

$$e^*_{(s,t)} = \min_{\mathcal{T}_{(s,t)}} \Big\{ P[b_k \in \mathcal{T}_{(s,t)}] e^*_{(s-1,t-1)} + P[b_k \in \mathcal{T}^c_{(s,t)}] e^*_{(s,t-1)} + P[b_k \in \mathcal{T}^c_{(s,t)}] - \max_{m_j \in \mathcal{T}^c_{(s,t)}} p_j \Big\}$$

or equivalently

$$e^*_{(s,t)} = \min_{\mathcal{T}^c_{(s,t)}} \Big\{ \Big(1 - P[b_k \in \mathcal{T}^c_{(s,t)}]\Big) e^*_{(s-1,t-1)} + P[b_k \in \mathcal{T}^c_{(s,t)}] e^*_{(s,t-1)} + P[b_k \in \mathcal{T}^c_{(s,t)}] - \max_{m_j \in \mathcal{T}^c_{(s,t)}} p_j \Big\}$$

Plugging in $P[b_k \in \mathcal{T}^c_{(s,t)}] = \sum_{m_i \in \mathcal{T}^c_{(s,t)}} p_i$, and rearranging the terms, we obtain the following error recursion:

$$e^*_{(s,t)} = e^*_{(s-1,t-1)} + \min_{\mathcal{T}^c_{(s,t)}} \Bigg\{ -(e^*_{(s-1,t-1)} - e^*_{(s,t-1)}) \sum_{m_i \in \mathcal{T}^c_{(s,t)}} p_i + \sum_{m_i \in \mathcal{T}^c_{(s,t)}} p_i - \max_{m_j \in \mathcal{T}^c_{(s,t)}} p_j \Bigg\} \tag{31}$$

We next show that the error difference, $e^*_{(s-1,t-1)} - e^*_{(s,t-1)}$, can be bounded from below and above.

Proposition 2. *Suppose* $1 \leq s \leq t$. *Then, the error difference* $e^*_{(s-1,t-1)} - e^*_{(s,t-1)}$ *satisfies:*

$$0 \leq e^*_{(s-1,t-1)} - e^*_{(s,t-1)} \leq 1$$

Proof. The lower bound can be established by observing that for the same decision horizon, $t-1$, the minimum average distortion achieved by s channel uses, is always at least as small as the one that can be achieved by $s-1$ channel uses. For the upper bound, one needs to observe that the maximum stage-wise estimation error is bounded by 1.

Using Proposition 2, we will next show that the optimum choice for the sets $\mathcal{T}^c_{(s,t)}$ is the singleton $\mathcal{T}^{c*}_{(s,t)} = \{m_{i^*}\}$, where $i^* = \arg\max_{m_i \in \mathcal{B}} p_i$.

In other words, the optimal solution is not to transmit the *most likely* outcome, and transmit all the other outcomes of the source process b_k. Moreover, this policy is independent of the number of decision instances left, t_k, and the number of transmission opportunities left, s_k, provided that $s_k \geq 1$. Recall that, the optimum estimator is the MAP estimator and is given by (21).

In order to show that this is indeed the optimal observer (or transmission) policy, we first set the cardinality of the set $\mathcal{T}_{(s,t)}$ to $|\mathcal{T}^c_{(s,t)}| = 0$, and determine that the expression inside the curly brackets in (31) is just 0.

We next set $|\mathcal{T}^c_{(s,t)}| = 1$, and note that the minimization of the function inside the curly brackets in (31) is equivalent to the following minimization:

$$\min_i \left\{(1-p_i)e^*_{(s-1,t-1)} + p_i e^*_{(s,t-1)} + p_i - p_i\right\}$$

Here i is such that $m_i \in \mathcal{B}$. Canceling p_i's and rearranging, we obtain an equivalent minimization problem:

$$\min_i -p_i(e^*_{(s-1,t-1)} - e^*_{(s,t-1)})$$

By Proposition 2, the error difference, $e^*_{(s-1,t-1)} - e^*_{(s,t-1)}$, is nonnegative; thus, the minimum is achieved by picking i as

$$i^* = \arg\max_{m_i \in \mathcal{B}} p_i$$

This choice yields a minimum value of $-(e^*_{(s-1,t-1)} - e^*_{(s,t-1)})p_{i^*}$. Note that this value is at least as good as the value we obtained when we set the cardinality of the set $|\mathcal{T}^c_{(s,t)}| = 0$. Thus, we never pick $\mathcal{T}^c_{(s,t)}$ such that it has zero cardinality.

Finally, we let $|\mathcal{T}^c_{(s,t)}| \geq 2$, and let $p_{\max}$ denote the element of $\mathcal{T}^c_{(s,t)}$ with the maximal probability. That is,

$$p_{\max} = \max_{m_j \in \mathcal{T}^c_{(s,t)}} p_j$$

Since the number of elements of $\mathcal{T}^c_{(s,t)}$ is at least 2, the minimization problem inside the curly brackets in (31) can be written as

$$\min_{\mathcal{T}^c_{(s,t)}} \left\{ -(e^*_{(s-1,t-1)} - e^*_{(s,t-1)})p_{\max} + (1 - (e^*_{(s-1,t-1)} - e^*_{(s,t-1)})) \sum_{m_i \in \mathcal{T}^c_{(s,t)} \setminus m_{j^*}} p_i \right\}$$

where

$$m_{j^*} = \arg \max_{m_i \in \mathcal{T}^c_{(s,t)}} p_i$$

Now, by Proposition 2, the term multiplying the sum $\sum_{m_i \in \mathcal{T}^c_{(s,t)} \setminus m_{j^*}} p_i$ is always nonnegative; hence, we can conclude that the above minimum is bounded from below by

$$-(e^*_{(s-1,t-1)} - e^*_{(s,t-1)})p_{\max}$$

for any choice of the set $\mathcal{T}^c_{(s,t)}$ with cardinality $|\mathcal{T}^c_{(s,t)}| \geq 2$. However the expression $-(e^*_{(s-1,t-1)} - e^*_{(s,t-1)})p_{\max}$ satisfies

$$-(e^*_{(s-1,t-1)} - e^*_{(s,t-1)})p_{\max} \geq -(e^*_{(s-1,t-1)} - e^*_{(s,t-1)})p_{i^*}$$

since $p_{i^*} \geq p_{\max}$. Therefore, the minimum value of the function inside the curly brackets in (31) is achieved when $\mathcal{T}^c_{(s,t)} = \{m_{i^*}\}$, as claimed.

In summary, when the distortion criterion is the probability of error, at time k, the optimal observer first observes the source output b_k. Then, it checks to see if $s_k \geq 1$; if so, it transmits b_k unless $b_k = m_{i^*}$, i.e., the most likely outcome. The estimator (or decoder), on the other hand, employs the MAP estimation rule given the output of the channel.

Gaussian Case

Suppose b_k's are zero-mean, i.i.d. Gaussian. Let $\Phi(\cdot)$ denote the cumulative density function (CDF) of the standard Gaussian random variable with zero mean and unit variance. In the Gaussian case, we can generalize our search for an optimum in (27) to more general intervals of the form $\mathcal{T}^c_{(s,t)} = [\alpha_{(s,t)}, \beta_{(s,t)}]$, where $-\infty \leq \alpha_{(s,t)} \leq \beta_{(s,t)} \ +\infty$.

Figure 6 shows the plot of the objective function on the right-hand side of (27) for the case when $\mathcal{T}_{(s,t)} = [a, b]$, $\sigma_b^2 = 1$, $e^*_{(s-1,t-1)} = 3$, and $e^*_{(s,t-1)} = 1$. Note that the minimum occurs at $b^* = -a^* = \sqrt{3-1} = \sqrt{2} = 1.4142$. Thus, even though we did not restrict ourselves to symmetric intervals, the solution is still a symmetric interval around zero. To show that this is indeed the case in general, one needs to differentiate the objective function inside the curly brackets in (27) with respect to both $\alpha_{(s,t)}$ and $\beta_{(s,t)}$, and show that the minimum occurs at $\beta^*_{(s,t)} = -\alpha^*_{(s,t)}$ when $f(b)$ is the Gaussian pdf [5].

To evaluate the optimum estimation error $e^*_{(s,t)}$ in terms of $e^*_{(s-1,t-1)}$ and $e^*_{(s,t-1)}$, we substitute the optimum interval solution (30) into the right-hand side of (27), and use the standard properties of the Gaussian density that we listed above to obtain

$$e^*_{(s,t)} = e^*_{(s-1,t-1)} - \left[e^*_{(s-1,t-1)} - e^*_{(s,t-1)} - \sigma_b^2\right] \times \left[2\Phi\left(\sqrt{\frac{e^*_{(s-1,t-1)} - e^*_{(s,t-1)}}{\sigma_b^2}}\right) - 1\right]$$

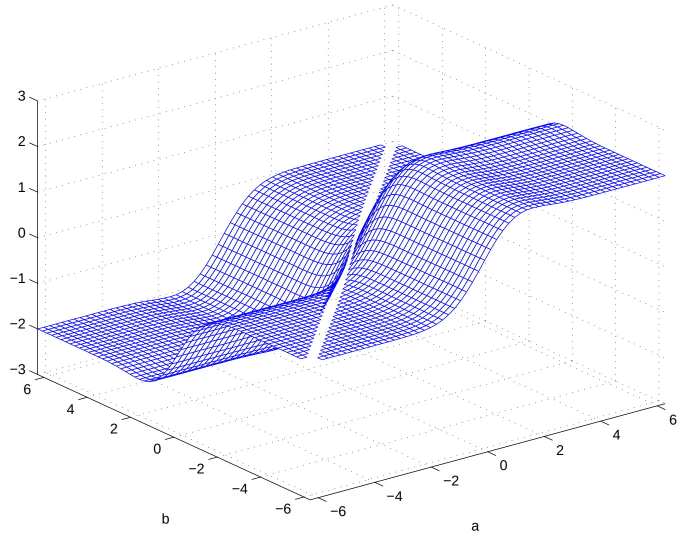

Fig. 6. Plot of the objective function in the Gaussian case with $\mathcal{T}_{(s,t)} = [a, b]$ when $\sigma_b^2 = 1$, $e^*_{(s-1,t-1)} = 3$, and $e^*_{(s,t-1)} = 1$.

$$-\frac{2\sigma_b^2}{\sqrt{2\pi\sigma_b^2}}\sqrt{e^*_{(s-1,t-1)} - e^*_{(s,t-1)}}\,e^{-\frac{e^*_{(s-1,t-1)} - e^*_{(s,t-1)}}{2\sigma_b^2}} \tag{32}$$

We can normalize the optimal estimation error by letting

$$\epsilon_{(s,t)} = \frac{e^*_{(s,t)}}{\sigma_b^2} \tag{33}$$

and rewrite the recursion (32) in a simpler form:

$$\begin{aligned}\epsilon_{(s,t)} = \epsilon_{(s-1,t-1)} &- \left[\epsilon_{(s-1,t-1)} - \epsilon_{(s,t-1)} - 1\right]\left[2\Phi\left(\sqrt{\epsilon_{(s-1,t-1)} - \epsilon_{(s,t-1)}}\right) - 1\right] \\ &- \frac{2}{\sqrt{2\pi}}\sqrt{\epsilon_{(s-1,t-1)} - \epsilon_{(s,t-1)}}\,e^{-\frac{\epsilon_{(s-1,t-1)} - \epsilon_{(s,t-1)}}{2}}\end{aligned} \tag{34}$$

with the initial conditions

$$\epsilon(t,t) = 0,\ \epsilon(0,t) = t, \quad \forall t \geq 0$$

and $\epsilon(s,t)$ is undefined for $s > t$.

Hence, we can provide a solution to the problem of optimal sequential estimation of an i.i.d. Gaussian process of finite length over a noiseless channel that can only be used a limited number of times. First, a table has to be formed by an offline numerical computation of the recursion (34). Then, the table can be scaled, if needed, to the actual variance of the process through (33). Next, the transmission intervals for the observer are determined via (29), and tabulated for all feasible pairs (s,t). For the online computation, as illustrated in Section 3.2, the observer has to keep two states, (s_k, t_k). Each time unit k, after observing the realization of the random variable b_k, the observer compares the realized value of the random variable to the optimum decision interval corresponding to the current state (s_k, t_k), and makes a transmission decision. The estimator, on the other hand, has access to the same tabulated values of the transmission intervals, $\mathcal{T}^*_{(s,t)}$, and it keeps track of the states (s_k, t_k) in the same way the observer does. Upon receiving the transmitted data, y_k, from the channel, the estimator simply applies the estimation policy given in Section 3.2.

Gaussian Case with Noisy Measurements

Let the source process b_k be i.i.d. Gaussian. If the observer has access to a noisy version of the source output, i.e.,

$$z_k = b_k + v_k$$

where v_k is zero-mean, i.i.d. Gaussian[20] with variance σ_v^2, the optimization problem with the mean-square distortion measure can be solved using a similar approach. In this case, the observer's decision as to whether to use the channel to transmit or not depends on the available data z_k. In the derivation of the optimal observer-estimator pair, most of the analysis of Section 3.2 carries over.

In order to see that the structure of the solution is preserved, first observe that when $s_k = 0$, $\hat{\mu}_k^* = 0$, and for $s_k \geq 1$, the optimal estimator has the form:

$$\hat{\mu}_k^*(I_k^d) = E\{b_k | (s_k, t_k); x_k\}$$

Substituting this into the error expression, and following along the lines of Proposition 1, one can see that the optimal observer policy has the form

$$\mu_k(I_k^e) = \bar{\mu}((s_k, t_k); z_k)$$

In other words $\{(s_k, t_k); z_k\}$ is a sufficient statistics for the optimal policy $\mu_k^*(I_k^e)$.

Since μ_k depends explicitly only on the current measurement z_k, for $s_k \geq 1$, the search for an optimal encoder policy can be narrowed down to the class of policies of the form

$$\mu_k(I_k^e) = \bar{\mu}((s_k, t_k); z_k) = \begin{cases} z_k & \text{if } z_k \in \mathcal{T}_{(s_k,t_k)} \\ \text{NT} & \text{if } z_k \in \mathcal{T}^c_{(s_k,t_k)} \end{cases} \tag{35}$$

where $\mathcal{T}_{(s_k,t_k)}$ is a measurable set on $\mathcal{B}$, and is a function of (s_k, t_k). Since $x_k = \mu_k(I_k^e)$, for $s_k \geq 1$, we can write the optimal estimator as

[20] We also assume that the processes $\{b_k\}$ and $\{v_k\}$ are independent.

$$\hat{\mu}_k((s_k,t_k);x_k) = \begin{cases} \frac{\sigma_b^2}{\sigma_b^2+\sigma_v^2} z_k & \text{if } z_k \in \mathcal{T}_{(s_k,t_k)} \\ E\left\{ b_k | z_k \in \mathcal{T}^c_{(s_k,t_k)} \right\} & \text{if } z_k \in \mathcal{T}^c_{(s_k,t_k)} \end{cases} \tag{36}$$

We proceed as in Section 3.2, and write the dynamic programming recursion governing the evolution of the optimal estimation error as follows:

$$e^*_{(s,t)} = \min_{\mathcal{T}^c_{(s,t)}} \left\{ e^*_{(s-1,t-1)} P[z \in \mathcal{T}_{(s,t)}] + \sigma_b^2 + \left(\frac{\sigma_b^2}{\sigma_b^2+\sigma_v^2} \right)^2 \int_{z \in \mathcal{T}_{(s,t)}} z^2 f_Z(z) dz \right.$$
$$\left. + e^*_{(s,t-1)} P[z \in \mathcal{T}^c_{(s,t)}] - 2 \frac{\sigma_b^2}{\sigma_b^2+\sigma_v^2} \int_{z \in \mathcal{T}_{(s,t)}} z E[b|z] f_Z(z) dz \right\}$$

where $f_Z(z) \sim N(0, \sigma_b^2+\sigma_v^2)$, and $f_{B|Z}(b|z) \sim N(\frac{\sigma_b^2}{\sigma_b^2+\sigma_v^2} z, \frac{\sigma_b^2\sigma_v^2}{\sigma_b^2+\sigma_v^2})$. The recursion can be simplified as

$$e^*_{(s,t)} = \min_{\mathcal{T}^c_{(s,t)}} \left\{ e^*_{(s-1,t-1)} + \sigma_b^2 - (e^*_{(s-1,t-1)} - e^*_{(s,t-1)}) \int_{z \in \mathcal{T}^c_{(s,t)}} f_Z(z) dz \right.$$
$$\left. - \left(\frac{\sigma_b^2}{\sigma_b^2+\sigma_v^2} \right)^2 (\sigma_b^2+\sigma_v^2) + \left(\frac{\sigma_b^2}{\sigma_b^2+\sigma_v^2} \right)^2 \int_{z \in \mathcal{T}^c_{(s,t)}} z^2 f_Z(z) dz \right\}$$

Following along the lines of Section 3.2, we restrict our search for an optimum set to simple intervals, i.e., $\mathcal{T}^c_{(s,t)} = [\alpha_{(s,t)}, \beta_{(s,t)}]$. The same analysis gives the optimum choice for $\beta_{(s,t)}$

$$\beta^*_{(s,t)} = \frac{\sigma_b^2+\sigma_v^2}{\sigma_b^2} \sqrt{e^*_{(s-1,t-1)} - e^*_{(s,t-1)}}$$

and $\alpha^*_{(s,t)} = -\beta^*_{(s,t)}$. Substituting these values into the error recursion, we obtain the two-dimensional recursion for the estimation error:

$$e^*_{(s,t)} = e^*_{(s-1,t-1)} - \left[e^*_{(s-1,t-1)} - e_{(s,t-1)} - \frac{(\sigma_b^2)^2}{\sigma_b^2+\sigma_v^2} \right]$$
$$\times \left[2\Phi \left(\frac{\sqrt{\sigma_b^2+\sigma_v^2}\sqrt{e^*_{(s-1,t-1)} - e^*_{(s,t-1)}}}{\sigma_b^2} \right) - 1 \right]$$
$$- \frac{2\sigma_b^2}{\sqrt{2\pi(\sigma_b^2+\sigma_v^2)}} \sqrt{e^*_{(s-1,t-1)} - e^*_{(s,t-1)}}\, e^{-\frac{\left(\frac{\sigma_b^2+\sigma_v^2}{\sigma_b^2}\right)^2 (e^*_{(s-1,t-1)} - e^*_{(s,t-1)})}{2(\sigma_b^2+\sigma_v^2)}}$$
$$+ \sigma_b^2 - \frac{(\sigma_b^2)^2}{\sigma_b^2+\sigma_v^2}$$

Note that, for $\sigma_v^2 = 0$ this recursion simplifies to (32), which is the recursion for the perfect state measurements.

We can normalize the optimal estimation error by letting

$$\epsilon_{(s,t)} = \frac{\sigma_b^2 + \sigma_v^2}{(\sigma_b^2)^2} e^*_{(s,t)} \tag{37}$$

and rewrite the above recursion in a simpler form:

$$\epsilon_{(s,t)} = \epsilon_{(s-1,t-1)} - \left[\epsilon_{(s-1,t-1)} - \epsilon_{(s,t-1)} - 1\right] \left[2\Phi\left(\sqrt{\epsilon_{(s-1,t-1)} - \epsilon_{(s,t-1)}}\right) - 1\right] - \frac{2}{\sqrt{2\pi}} \sqrt{\epsilon_{(s-1,t-1)} - \epsilon_{(s,t-1)}} e^{-\frac{\epsilon_{(s-1,t-1)} - \epsilon_{(s,t-1)}}{2}} + \frac{\sigma_v^2}{\sigma_b^2} \tag{38}$$

with the initial conditions

$$\epsilon(t,t) = \frac{\sigma_v^2}{\sigma_b^2} t, \ \epsilon(0,t) = \left(1 + \frac{\sigma_v^2}{\sigma_b^2}\right) t, \quad \forall t \geq 0$$

and $\epsilon(s,t)$ is undefined for $s > t$.

We note that the recursion (38) reduces to the recursion (34), as the noise variance $\sigma_v^2 \to 0$.

3.3 Estimating a Gauss-Markov Process with Limited Measurements

In this section, we discuss the case when the source process is Markov

$$b_{k+1} = A b_k + w_k$$

driven by an i.i.d. Gaussian process $\{w_k\}$ with zero-mean. The solution to this case is similar to the Gaussian i.i.d. case when the observer has access to the source output b_k without noise. The only difference is that, now the observer-estimator pair has to keep track of three variables (r_k, s_k, t_k), where r_k keeps track of the number of time units passed since the last use of the channel for transmission. A similar DP recursion, now in three dimensions, can be obtained.

Let r denote the number of time units passed since the last transmission of a source output. Reasoning as in Section 3.2, we can deduce that for $s \geq 1$, the optimal estimator has the form

$$\hat{\mu}((r,s,t); b_{N-t}) = \begin{cases} b_{N-t} & b_{N-t} \in \mathcal{T}_{(r,s,t)} \\ E\left\{b_{N-t} | b_{N-t} \in \mathcal{T}^c_{(r,s,t)}\right\} & b_{N-t} \in \mathcal{T}^c_{(r,s,t)} \end{cases} \tag{39}$$

With the estimator structure in place, the error recursion can be derived following along the lines of previous sections:

$$e^*_{(r,s,t)} = \min_{\mathcal{T}_{(r,s,t)}} \left\{ e^*_{(1,s-1,t-1)} P[b_{N-t} \in \mathcal{T}_{(r,s,t)}] + e^*_{(r+1,s,t-1)} P[b_{N-t} \in \mathcal{T}^C_{(r,s,t)}] + \int_{b_{N-t} \in \mathcal{T}^c_{(r,s,t)}} \left[b_{N-t} - A^r b_{N-t-r}\right]^2 f_{b_{N-t}}(b_{N-t}) db_{N-t} \right\}$$

where $b_{N-t} \sim N(A^r b_{N-t-r}, \left(\sum_{k=1}^{r} A^{2(k-1)}\right) \sigma_b^2)$.

Now if we let $\mathcal{T}^c_{(r,s,t)} = [\alpha_{(r,s,t)} \; \beta_{(r,s,t)}]$, the optimal choices for the parameters $\alpha_{(r,s,t)}$ and $\beta_{(r,s,t)}$ are

$$\alpha^*_{(r,s,t)} = A^r b_{N-t-r} + \sqrt{e^*_{(1,s-1,t-1)} - e^*_{(r+1,s,t-1)}}$$
$$\beta^*_{(r,s,t)} = A^r b_{N-t-r} - \sqrt{e^*_{(1,s-1,t-1)} - e^*_{(r+1,s,t-1)}}$$

Substituting these choices back into the error recursion and simplifying yields

$$\begin{aligned} e^*_{(r,s,t)} = e^*(1,s-1,t-1) - &\left[e^*_{(1,s-1,t-1)} - e^*_{(r+1,s,t-1)} - \left(\sum_{k=1}^{r} A^{2(k-1)}\right)\sigma_b^2\right] \\ &\times \left[2\Phi\left(\sqrt{\frac{e^*_{(1,s-1,t-1)} - e^*_{(r+1,s,t-1)}}{\sum_{k=1}^{r} A^{2(k-1)}\sigma_b^2}}\right) - 1\right] \\ &- \frac{2\sqrt{\sum_{k=1}^{r} A^{2(k-1)}\sigma_b^2}}{2\pi}\sqrt{e^*_{(1,s-1,t-1)} - e^*_{(r+1,s,t-1)}}\, e^{-\frac{e^*_{(1,s-1,t-1)} - e^*_{(r+1,s,t-1)}}{2\sum_{k=1}^{r} A^{2(k-1)}\sigma_b^2}} \end{aligned} \tag{40}$$

where we have made use of the fact that for a Gaussian random variable x with mean m and variance k^2, and for $a \geq 0$, we have the expression

$$\int_{m-\sqrt{a}}^{m+\sqrt{a}} (x-m)^2 f_x(x)dx = -\frac{2\sqrt{a}k}{\sqrt{2\pi}} e^{-\frac{a}{2k^2}} + k^2\left(2\Phi\left(\frac{\sqrt{a}}{k}\right) - 1\right)$$

The recursion (40) is defined for $r \geq 1$, and $0 \leq s \leq t$ with the boundary conditions given by

$$e^*_{(r,t,t)} = 0,\; e^*_{(r,0,t)} = \left(\sum_{l=r}^{r+t-1}\sum_{k=1}^{l} A^{2(k-1)}\right)\sigma_b^2 \tag{41}$$

Note that, as in Section 3.2, one can define the normalized estimation error by

$$\epsilon_{(r,s,t)} = \frac{1}{\left(\sum_{k=1}^{r} A^{2(k-1)}\right)\sigma_b^2} e^*_{(r,s,t)} \tag{42}$$

and simplify the recursion (40) further as follows:

$$\begin{aligned} \epsilon_{(r,s,t)} = \epsilon_{(1,s-1,t-1)} - &\left[\epsilon_{(1,s-1,t-1)} - \epsilon_{(r+1,s,t-1)} - 1\right] \\ &\times \left[2\Phi\left(\sqrt{\epsilon_{(1,s-1,t-1)} - \epsilon_{(r+1,s,t-1)}}\right) - 1\right] \\ &- \frac{2}{\sqrt{2\pi}}\sqrt{\epsilon_{(1,s-1,t-1)} - \epsilon_{(r+1,s,t-1)}}\, e^{-\frac{\epsilon_{(1,s-1,t-1)} - \epsilon_{(r+1,s,t-1)}}{2}} \end{aligned}$$

Note that when $r = 0$, this is the exact same recursion as in the case of estimating an i.i.d. Gaussian process with no measurement noise. The only difference between this case and the i.i.d. case is in scaling back into the original estimation error via (42). However, unlike the i.i.d. case, this recursion must be solved offline for all feasible (r, s, t) triplets, and a three-dimensional table has to be formed.

3.4 Illustrative Examples

Example 1

As an example for the case when the source is binary, i.e., $b_k \in \{0, 1\}$, consider the problem of sequentially estimating a Bernoulli process of length N with M opportunities to transmit over a noiseless binary channel. This problem is a special case of the general problem we solved in Section 3.2. The probability distribution of the source is given, and say, without loss of any generality, that 1 is a more likely outcome than 0. In this case, the best observation policy is to start at time $k = 0$ not transmit the likely outcome 1, and to use the channel to transmit only the unlikely outcome 0. And the best estimation scheme is to employ the MAP estimator which estimates NT as 1, and 0 as 0, as long as $s_k \geq 1$. If $s_k = 0$, on the other hand, then the best estimator should estimate 1 regardless of the channel output.

Example 2

The second example is just solving the problem of Section 3.2 for $(s, t) = (1, 2)$. So, the observer can use the channel for transmission only once, at time $k = 0$ or 1, and the observer and the estimator are jointly trying to minimize the average distortion (or estimation error):

$$e = E\left\{(b_0 - \hat{b}_0)^2 + (b_1 - \hat{b}_1)^2\right\}$$

where b_0, b_1 are i.i.d. Gaussian with zero mean, and variance σ_b^2. If we arbitrarily choose to transmit the first source output, or the second one, the estimation error would be

$$e^*_{no-observer} = \sigma_b^2$$

which is the best error that can be achieved without a decision maker that observes the source output. Now, suppose the observer is aware of the fact that the estimator knows the *a priori* distribution of b_0. So, it makes sense for the observer not to transmit the realized value of b_0 if this value happens to be close to the *a priori* estimate of it, which in this case is the mean value of b_0, i.e., zero.

Motivated by this intuition, the observer decides to adopt a policy in which it will not use the channel to transmit b_0 if it lies in an interval $[\alpha, \beta]$ around zero. Note that the decision for the second stage would already have been made once α and β are determined, because, if $b_0 \in [\alpha, \beta]$, then the observer cannot use the channel to transmit at time 1, and if $b_0 \notin [\alpha, \beta]$, there is no reason why it should not transmit at time 1.

Now, the optimization problem faced by the observer is to choose α and β such that the following error is minimized:

$$e_{(\alpha,\beta)} = \int_{\alpha}^{\beta} \left(b - E\left\{b|b \in [\alpha,\beta]\right\}\right)^2 f(b)db + \sigma_b^2 P\left\{b_0 \notin [\alpha,\beta]\right\}$$

where $f(b)$ is the standard Gaussian density. The solution can be easily obtained by checking the first and second order optimality conditions, and is given by

$$(\alpha^*, \beta^*) = (-\sigma_b, \sigma_b)$$

Thus, the observer should not use the channel to transmit the source output b_0 if it falls within one standard deviation of its mean. For these values of α and β, the optimal value of the estimation error can be calculated as

$$e_{(\alpha^*,\beta^*)} = \sigma_b^2 \left[1 - \sqrt{\frac{2}{\pi e}}\right] \tag{43}$$

Comparing this error to the no-observer policy, $e^*_{no-observer} = \sigma_b^2$, we see that there is an approximately $\sqrt{\frac{2}{\pi e}} \approx 48\%$ improvement in the estimation error.

Example 3

The third and final example we will discuss considers the following design problem. We are given a time-horizon of a fixed length N, say 100 For this $N = 100$ time units, we would like to sequentially estimate the state of a zero-mean, i.i.d. Gaussian process with unit variance. We have a design criterion which says that the *aggregate* estimation error should not exceed 20. The solution to this problem without an observer agent is to reveal 80 arbitrary observations to the estimator and achieve an aggregate estimation error of 20. Suppose, now we use the optimal observer-estimator pair. In Figure 7, we plot the optimal value of the 100-stage estimation error for different values of M.

It is striking that a cumulative estimation error of 20 can be achieved with only 34 transmissions. This is approximately a $\frac{80-34}{80} \times 100 \approx 58\%$ improvement over the no-observer policy.

In order to verify our design, we simulate the optimal observer and estimator policies in Matlab. Figure 8 shows a typical sample path of the optimal number of channel uses left for a decision horizon of length $N = 100$, and a limited, $M = 34$, number of channel uses. The sample paths depend on the realization of the random sequence $\{b_k\}_0^{N-1}$.

4 Conclusions

In this paper, we introduced some new hard-constrained sequential estimation problems with applications in wireless sensing. We showed that the problems can be solved using dynamic-programming type arguments, and their solutions have a threshold characterization. The process models considered in this paper

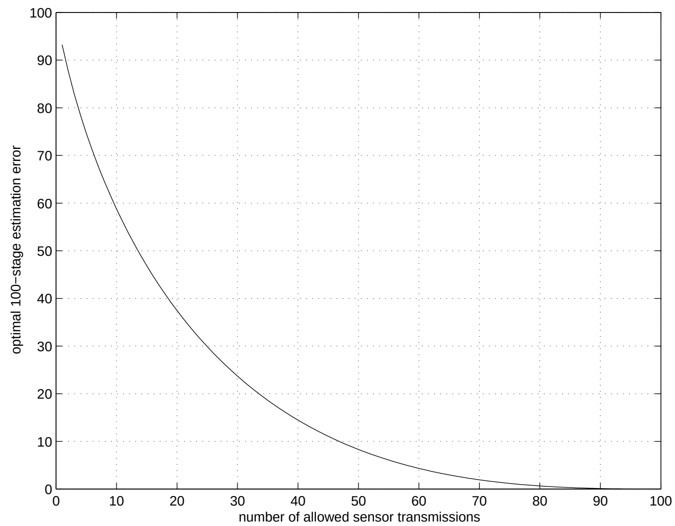

Fig. 7. Optimal 100-stage estimation error vs. the number of allowed channel uses.

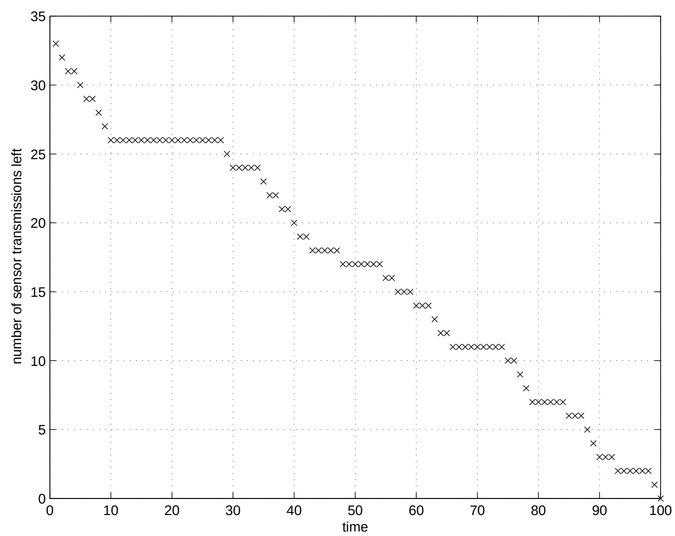

Fig. 8. A typical sample path of the number of channel uses left under the optimal observer-estimator policies. $(N, M) = (100, 34)$.

were idealized for ease of presentation and mathematical tractability. However, the basic thinking behind these models can be easily adopted to real-world wireless sensing problems with power constraints. When doing so, one needs to consider several other design requirements imposed on the system, such as network-level connectivity and time synchronization. Current research effort is directed towards developing algorithms that take some of these cross-layer design issues into account.

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