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Dynamic Optimization Flow Control

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Abstract—**In this paper, we introduce a class of flow control problems in a network which involves dynamic optimization. As opposed to the static flow control problem where the source rates are determined as a result of a static optimization problem, in our formulation the rates are adjusted according to the solution of an infinite-horizon discounted optimal control problem. The objective is to maximize the aggregate source utility over transmission rates, while at the same time keeping the buffer occupancies throughout the network at an acceptable level.**

I. Introduction

Optimization based flow control is based on the premise that sources with different valuation of. bandwidth should react differently to network congestion. One way of quantifying these different valuations of the sources is through utility functions. The utility function of a source basically tells us how much enjoyment the source can get out of transmitting at a given rate. The classical approach to optimization based flow control has the objective of maximizing the aggregate source utility over their transmission rates. This approach has been extensively studied in several papers, such as [1], [2], [3], [4], [5], [6], [7]. In all these works, the optimization problem that determines the operating point is a static one in that the transmission rate of each source is determined so as to maximize the total utility of all sources subject to link capacity constraints. Solving this static optimization problem centrally would require not only the knowledge of all utility functions, but also complex coordination among potentially all sources due to coupling of sources through shared links. Thus, the goal is to find a decentralized algorithm in which sources and links have limited amount of communication. Most of the previous research has been directed towards finding such algorithms that converge to the solution of the static optimization problem.

Note that in all these approaches even though one starts with a static maximization problem, the way sources reach the solution of this maximization problem is dynamic. Since the behavior of the network is inherently dynamic towards any decentralized optimization, in this paper we raise the question: "Why not start with a dynamic optimization problem in the first place?". Formulating network flow problems as dynamic control problems has in fact been done before. For example, in [8], [9], [10] the ATM ABR congestion control problem was formulated as a stochastic team problem. However, this paper is the first attempt to use a dynamic optimization approach for utility-based flow control problems.

In order to bring in dynamics into the problem, we consider the buffer occupancy of each link in the network, and model its behavior as a discrete-time dynamic system. The source rates are inputs to this system. To formulate a dynamic optimization problem, we combine the objective of clearing the buffers with the objective of the static optimization problem of maximizing the aggregate source utility over their transmission rates. These two objectives are reflected in an infinite-horizon discounted cost function that is to be maximized over a feasible set of source rates. The solution to this optimization problem is obtained by solving the associated Bellman's equation, which leads to a centralized solution, which is not appealing at first sight. However, there do exist decentralized algorithms, as in the case of the static optimization flow control problem, that achieve the optimum solution in the limit. We characterize these decentralized algorithms, one of which is very similar to REM (Random Exponential Marking). Its convergence properties are still under investigation.

This paper is organized as follows. In Section 2, we describe the dynamic optimization problem, and in Section 3 we obtain its solution. Section 4 is devoted to a general discussion on the implications of the solution, as well as some decentralized algorithms to implement the solution on a network. The paper ends with the concluding remarks of Section 5, in which we also discuss some future research directions.

II. The Dynamic Optimization Problem

Consider a network that consists of a set $\mathcal{L} = \{1, \ldots, L\}$ of links of capacity c_l, $l \in \mathcal{L}$. The network is shared by a set $\mathcal{S} = \{1, \ldots, S\}$ of sources. Source s transmits at rate x_s using a set $\mathcal{L}_s \subseteq \mathcal{L}$ of links. The rate x_s satisfies $m_s \leq x_s \leq M_s$, where $m_s \geq 0$ and $M_s < \infty$ are the minimum and maximum transmission rates, respectively. When transmitting at rate x_s, source

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s attains a utility of $U_s(x_s)$. It is assumed that the utility function U_s is increasing and strictly concave in its argument. For each link l, let $\mathcal{S}_l$ denote the set of sources that use link l. Associated with each link l there is a buffer with occupancy b_l. The buffer length b_l satisfies $0 \leq b_l \leq B_l$, where $B_l < \infty$ is the maximum buffer occupancy.

We assume an underlying discrete time structure with no delay and synchronous updates. Let t denote the discrete time unit, and assume that fluid approximation for queue lengths holds. Then, the buffer occupancy $b_l(t)$ at link l at time t evolves according to

$$b_l(t+1) = \left[b_l(t) + \sum_{s \in \mathcal{S}_l} x_s(t) - c_l \right]_0^{B_l} \qquad (1)$$

where the notation $[x]_{x_{\min}}^{x_{\max}}$ is shorthand for

$$[x]_{x_{\min}}^{x_{\max}} = \begin{cases} x_{\min} & x \leq x_{min} \\ x & x_{\min} \leq x \leq x_{\max} \\ x_{\max} & x \geq x_{\max} \end{cases}$$

We assume that $b_l(0)$, $l \in \mathcal{L}$, is known.

Define the state space $\mathcal{B} := \{(b_1, \ldots, b_L) \in \mathcal{R}^L \mid 0 \leq b_l \leq B_l, \ l = 1, \ldots, L\}$, the control space $\mathcal{X} := \{(x_1, \ldots, x_S) \in \mathcal{R}^S \mid m_s \leq x_s \leq M_s, \ s = 1, \ldots, S\}$, and the vectors $b(t) := [b_1(t) \ \cdots \ b_L(t)]^T$ and $x(t) := [x_1(t) \ \cdots \ x_S(t)]^T$.

Given an initial state $b(0)$, our objective is to find a *stationary* policy u, where $u : \mathcal{B} \to \mathcal{X}$, that maximizes the discounted infinite horizon cost function[1]

$$J_u(b(0)) = \lim_{T \to \infty} \sum_{t=0}^{T-1} \alpha^t \left[\sum_{s \in \mathcal{S}} U_s(u_s(b(t))) - \sum_{l \in \mathcal{L}} \beta_l p_l(b(t+1)) \right] \qquad (2)$$

subject to the system equation constraint (1). Here, $0 < \alpha < 1$ is the *discount factor*, β_l, $l \in \mathcal{L}$ are positive constants, and $p_l(\cdot)$, $l \in \mathcal{L}$ are *penalty* functions. We assume that p_l is increasing and strictly convex in its argument, with $p_l(0) = 0$ and $p_l'(0) > 0$.

The objective function (2) has two components. The sum of utilities of all sources, and a penalty for high buffer occupancy. Hence, it is designed to achieve maximal social utility while at the same time keeping the buffer lengths under control.

We denote by U the set of all admissible policies, that is the set of all functions u with $u : \mathcal{B} \to \mathcal{X}$. The optimal cost function J^* is defined by

$$J^*(b) = \sup_{u \in U} J_u(b), \ b \in \mathcal{B}$$

[1] In what follows, we will impose assumptions on the cost per stage and the discount factor α that guarantee that this limit exists.

In the next section, we show that under certain assumptions, the optimal cost function J^* satisfies Bellman's equation, and further show how one can solve this equation under some additional assumptions.

III. Bellman's Equation and its Solution

We start by showing that the cost per stage in (2) is bounded.

Lemma 3.1 (Boundedness): *The cost per stage satisfies*

$$\left| \sum_{s \in \mathcal{S}} U_s(x_s) - \sum_{l \in \mathcal{L}} \beta_l p_l([b_l + \sum_{s \in \mathcal{S}_l} x_s - c_l]_0^{B_l}) \right| \leq \kappa,$$

$\forall (b, x) \in \mathcal{B} \times \mathcal{X}$ *where* $\kappa < \infty$ *is some positive scalar.*

Proof: This follows from the boundedness of the sets $\mathcal{B}$ and $\mathcal{X}$, and monotonicity of the utility as well as penalty functions. We have

$$\begin{aligned} & \sum_{s \in \mathcal{S}} U_s(m_s) - \sum_{l \in \mathcal{L}} \beta_l p_l(B_l) \\ \leq & \sum_{s \in \mathcal{S}} U_s(x_s) - \sum_{l \in \mathcal{L}} \beta_l p_l([b_l + \sum_{s \in \mathcal{S}_l} x_s - c_l]_0^{B_l}) \\ \leq & \sum_{s \in \mathcal{S}} U_s(M_s) \end{aligned}$$

Let

$$\kappa = \max \left\{ \left| \sum_{s \in \mathcal{S}} U_s(m_s) - \sum_{l \in \mathcal{L}} \beta_l p_l(B_l) \right|, \sum_{s \in \mathcal{S}} U_s(M_s) \right\}$$

which completes the proof of the Lemma. □

It is well known that if the cost per stage in an infinite-horizon discounted optimal control problem is bounded (Lemma 3.1), and furthermore $0 < \alpha < 1$, then the optimal cost function J^* satisfies [11]

$$\begin{aligned} J^*(b) = \max_{x \in \mathcal{X}} \Big\{ & \sum_{s \in \mathcal{S}} U_s(x_s) \\ & - \sum_{l \in \mathcal{L}} \beta_l p_l([b_l + \sum_{s \in \mathcal{S}_l} x_s - c_l]_0^{B_l}) \\ & + \alpha J^*([b + \sum x_s - c]_0^{B}) \Big\}, \end{aligned} \qquad (3)$$

$\forall b \in \mathcal{B}$, which is *Bellman's equation*. In (3), $J^*([b + \sum x_s - c]_0^B)$ should actually read

$$J^*([b_1 + \sum_{s \in \mathcal{S}_1} x_s - c_1]_0^{B_1}, \ldots, [b_L + \sum_{s \in \mathcal{S}_L} x_s - c_L]_0^{B_L})$$

The optimal cost function J^* is the unique solution of this equation within the class of bounded functions. Furthermore, a stationary policy u is optimal if and only if $u(b)$ attains the maximum in Bellman's equation (3) for each $b \in \mathcal{B}$.

Before solving (3), we state a rather obvious result that would simplify the solution process.

Lemma 3.2 (Scaling): *For each link l, the maximum buffer occupancy B_l can be picked without any loss of generality to satisfy*

$$B_l \leq c_l - \sum_{s \in S_l} m_s \tag{4}$$

Proof: For each link l, $\exists t_l > 0$ such that

$$t_l \geq \frac{B_l}{c_l - \sum_{s \in S_l} m_s}$$

Now, pick the discrete time unit t so that

$$t \geq \max_{l \in \mathcal{L}} t_l,$$

and this concludes the proof. □

Lemma 3.2 simply asserts that by adjusting the discrete time unit, i.e. the update frequency of the fluid model, one can make sure that (4) holds. For example, say a link in the network has a capacity of 100 Mbits/second, and the maximum size of queueing memory (buffer) is 64 Kbytes. Then, by sampling at a frequency no smaller than 5.12 miliseconds, one can satisfy (4) for this particular link.

Next, for a given $b \in \mathcal{B}$ consider the following partitioning of the set $\mathcal{X}$:

$$\mathcal{X}_+(b) = \{x \in \mathcal{X} \mid \sum_{s \in S_l} x_s \leq c_l - b_l, \text{ for all } l \in \mathcal{L}\}$$

$$\mathcal{X}_-(b) = \{x \in \mathcal{X} \mid \sum_{s \in S_l} x_s \geq c_l - b_l, \text{ for some } l \in \mathcal{L}\}$$

Note that $\mathcal{X}_+(b) \cup \mathcal{X}_-(b) = \mathcal{X}$. Next two lemmas show that $\mathcal{X}_+(b)$ is nonempty, and monotonic in b.

Lemma 3.3 (Nonemptiness): *For a given $b \in \mathcal{B}$, the set $\mathcal{X}_+(b)$ is nonempty.*

Proof: By Lemma 3.2, for each $l \in \mathcal{L}$

$$c_l - b_l \geq c_l - B_l \geq \sum_{s \in S_l} m_s$$

Also, for each $x \in \mathcal{X}_+(b)$

$$\sum_{s \in S_l} m_s \leq \sum_{s \in S_l} x_s \tag{5}$$

for all $l \in \mathcal{L}$. Combining the two inequalities, we see that $m = (m_s, s \in \mathcal{S}) \in \mathcal{X}_+(b)$. Hence, $\mathcal{X}_+(b)$ is nonempty. □

Lemma 3.4 (Monotonicity): *Given $b^1, b^2 \in \mathcal{B}$, $\mathcal{X}_+(b^1) \subseteq \mathcal{X}_+(b^2)$ if $b^2 \leq b^1$, where the inequality is componentwise.*

Proof: The proof follows directly from the definition of $\mathcal{X}_+(b)$. □

In order to obtain the solution of Bellman's equation, we break the maximization problem in the right hand side of (3) over two subproblems over the sets $\mathcal{X}_+(b)$ and $\mathcal{X}_-(b)$. Then, the overall maximum is obtained by taking the larger of the two maximums. Lemma 3.5, which we give without a proof, justifies this operation.

Lemma 3.5: *Consider the maximization problem $\max_{x \in \mathcal{X}} g(x)$. Let $\mathcal{X}_+$ and $\mathcal{X}_-$ be such that $\mathcal{X}_+ \cup \mathcal{X}_- = \mathcal{X}$. Suppose $\max_{x \in \mathcal{X}_-} g(x) \leq \max_{x \in \mathcal{X}_+} g(x)$. Then,*

$$\max_{x \in \mathcal{X}} g(x) = \max_{x \in \mathcal{X}_+} g(x)$$

Applying Lemma 3.5 to (3), we first would like to solve the following equation:

$$\begin{aligned} J^+(b) &= \max_{x \in \mathcal{X}_+(b)} \Big\{ \sum_{s \in \mathcal{S}} U_s(x_s) \\ &\quad - \sum_{l \in \mathcal{L}} \beta_l p_l([b_l + \sum_{s \in S_l} x_s - c_l]_0^{B_l}) \\ &\quad + \alpha J^+([b + \sum x_s - c]_0^B) \Big\} \end{aligned} \tag{6}$$

for a given $b \in \mathcal{B}$. Since $x \in \mathcal{X}_+(b)$, we have $\sum_{s \in S_l} x_s \leq c_l - b_l$, which implies

$$[b_l + \sum_{s \in S_l} x_s - c_l]_0^{B_l} = 0, \ \forall l \in \mathcal{L}$$

Thus, (6) simplifies to

$$J^+(b) = \max_{x \in \mathcal{X}_+(b)} \left\{ \sum_{s \in \mathcal{S}} U_s(x_s) \right\} + \alpha J^+(0) \tag{7}$$

as $p_l(0) = 0$. The initial condition $J^+(0)$ can be found to be

$$J^+(0) = \frac{1}{1-\alpha} \max_{x \in \mathcal{X}_+(0)} \left\{ \sum_{s \in \mathcal{S}} U_s(x_s) \right\}$$

By Lemma 3.3, we know that $\mathcal{X}_+(b)$ is nonempty for each $b \in \mathcal{B}$. Therefore, the maximization in (7) is well-defined for all $b \in \mathcal{B}$. As a result, (7) defines $J^+(b)$ for all $b \in \mathcal{B}$. In what follows we will show that $J^+(b)$ is indeed $J^*(b)$, the optimal cost function.

Note that the stationary policies that maximize the right hand side of (7) are the solution of the following maximization problem:

$$\max_{x \in \mathcal{X}} \sum_{s \in \mathcal{S}} U_s(x_s) \tag{8}$$

$$\text{subject to} \quad \sum_{s \in S_l} x_s \leq c_l - b_l, \ l = 1, \ldots, L \tag{9}$$

This is nothing but the *primal problem* for the static optimization flow control problem, with link capacities c_l replaced by $c_l - b_l$. Hence, its solution can be found in a decentralized way using either the primal or the dual algorithm [1], [2], [3], [4]. We have more on this in the next section.

We will need the following Lemma in the sequel.

Lemma 3.6: $J^+(b)$ *has the property*

$$\frac{\partial J^+(b)}{\partial b_l} \le 0$$

for all $b \in \mathcal{B}$, $l \in \mathcal{L}$.

Proof: Pick any point $b \in \mathcal{B}$. Let $b^{l,h} = (b_1, \dots, b_l + h, \dots, b_L)$, and consider the limit

$$\frac{\partial J^+(b)}{\partial b_l} = \lim_{h \to 0} \frac{J^+(b^{l,h}) - J^+(b)}{h}$$

which equals

$$\lim_{h\to 0} \frac{\max_{x\in\mathcal{X}_+(b^{l,h})} \sum U_s(x_s) - \max_{x\in\mathcal{X}_+(b)} \sum U_s(x_s)}{h}$$

The denominator of the fraction in the limit is less than or equal to zero from Lemma 3.4, since $b \le b^{l,h}$. □

Let $\mathcal{L}_- \subseteq \mathcal{L}$ be the set of links for which

$$\sum_{s\in\mathcal{S}_l} x_s \ge c_l - b_l$$

Note that for a given $b \in \mathcal{B}$, the set $\mathcal{L}_-$ is not fixed as x varies over $\mathcal{X}_-(b)$. The set $\mathcal{L}_-$ can be written as the disjoint union of the sets: $\mathcal{L}_-^- = \{l \in \mathcal{L}_- \mid b_l + \sum_{s\in\mathcal{S}_l} x_s - c_l \le B_l\}$ and $\mathcal{L}_-^+ = \{l \in \mathcal{L}_- \mid b_l + \sum_{s\in\mathcal{S}_l} x_s - c_l > B_l\}$.

Now, for a given set of links $\mathcal{L}_-$, define $J^{\mathcal{L}_-}(b)$ as

$$\begin{aligned} J^{\mathcal{L}_-}(b) &= \max_{x\in\mathcal{X}_{\mathcal{L}_-}(b)} \Big\{ \sum_{s\in\mathcal{S}} U_s(x_s) \\ &\quad - \sum_{l\in\mathcal{L}} \beta_l p_l([b_l + \sum_{s\in\mathcal{S}_l} x_s - c_l]_0^{B_l}) \\ &\quad + \alpha J^{\mathcal{L}_-}([b + \sum x_s - c]_0^B) \Big\} \\ &= \max_{x\in\mathcal{X}_{\mathcal{L}_-}(b)} \Big\{ \sum_{s\in\mathcal{S}} U_s(x_s) \\ &\quad - \sum_{l\in\mathcal{L}_-^-} \beta_l p_l(b_l + \sum_{s\in\mathcal{S}_l} x_s - c_l) \\ &\quad - \sum_{l\in\mathcal{L}_-^+} \beta_l p_l(B_l) - \sum_{l\in\mathcal{L}_+} \beta_l p_l(0) \\ &\quad + \alpha J^{\mathcal{L}_-}(b + \sum x_s - c) \Big\} \end{aligned} \tag{10}$$

where $\mathcal{X}_{\mathcal{L}_-}(b) \subseteq \mathcal{X}_-$ is defined by

$$\mathcal{X}_{\mathcal{L}_-}(b) = \Big\{ x \in \mathcal{X} \mid \sum_{s\in\mathcal{S}_l} x_s \ge c_l - b_l,\ l \in \mathcal{L}_-, \sum_{s\in\mathcal{S}_l} x_s \le c_l - b_l,\ l \in \mathcal{L} \setminus \mathcal{L}_- \Big\}$$

In (10), $J^{\mathcal{L}_-}(b + \sum x_s - c)$ should read

$$J^{\mathcal{L}_-}(1_{\mathcal{L}_-}(l)[b_l + \sum_{s\in\mathcal{S}_l} x_s - c_l]^{B_l})$$

where $1_{\mathcal{L}_-}(l)$ is the indicator function of the set $\mathcal{L}_-$.

Now, we are in a position to prove the main Theorem of this section that gives the solution of Bellman's equation (3) and the stationary policy that maximizes the cost (2).

Theorem 3.1: *Let* $\beta_l = \beta$ *for each link* $l \in \mathcal{L}$. *If* β *satisfies*

$$\beta \ge \frac{U_s'(m_s)}{p_s'(0)},\ \forall s \in \mathcal{S} \tag{11}$$

then $J^+(b)$ *solves Bellman's equation (3) for all* $b \in \mathcal{B}$. *Furthermore, the solution of the maximization problem (8)-(9) gives the stationary policy that maximizes the infinite-horizon discounted cost (2).*

Proof: To show that $J^+(b)$ solves Bellman's equation we need to prove that if $J^{\mathcal{L}_-}(b) \equiv J^+(b)$ for all sets $\mathcal{L}_- \subseteq \mathcal{L}$, then

$$\begin{aligned} &\max_{x\in\mathcal{X}_{\mathcal{L}_-}(b)} \Big\{ \sum_{s\in\mathcal{S}} U_s(x_s) \\ &\quad - \sum_{l\in\mathcal{L}_-^-} \beta_l p_l(b_l + \sum_{s\in\mathcal{S}_l} x_s - c_l) \\ &\quad - \sum_{l\in\mathcal{L}_-^+} \beta_l p_l(B_l) \\ &\quad + \alpha J^+(1_{\mathcal{L}_-}(l)[b_l + \sum_{s\in\mathcal{S}_l} x_s - c_l]^{B_l}) \Big\} \\ &\le J^+(b) \end{aligned} \tag{12}$$

for all $b \in \mathcal{B}$. If (12) holds, then by Lemma 3.5, we have $J^*(b) = J^+(b)$. Since solution of (8)-(9) maximizes $J^+(b)$, the stationary policy that maximizes $J^*(b)$ would be given by (8)-(9). We proceed by showing that inequality (12) indeed holds.

For ease of notation, let

$$\begin{aligned} V(x) &= \sum_{s\in\mathcal{S}} U_s(x_s) - \sum_{l\in\mathcal{L}_-^-} \beta_l p_l(b_l + \sum_{s\in\mathcal{S}_l} x_s - c_l) \\ &\quad - \sum_{l\in\mathcal{L}_-^+} \beta_l p_l(B_l) \\ &\quad + \alpha J^+(1_{\mathcal{L}_-}(l)[b_l + \sum_{s\in\mathcal{S}_l} x_s - c_l]^{B_l}) \end{aligned}$$

Thus, equation (12) can be written as

$$\max_{x\in\mathcal{X}_{\mathcal{L}_-}(b)} V(x) \le J^+(b) \tag{13}$$

First, observe that at $\sum_{s\in\mathcal{S}_l} x_s = b_l - c_l$, $\max_{x\in\mathcal{X}_{\mathcal{L}_-}(b)} V(x) = J^+(b)$. Clearly, if we can show that for all $x \in \mathcal{X}_{\mathcal{L}_-}(b)$ $V(x)$ is decreasing, we will be able to conclude that (13) holds. To this end, for each link $l \in \mathcal{L}_-$, define the unit vectors d^l by

$$d^l = \frac{1}{\sqrt{|\mathcal{S}_l|}}(d_1^l, \dots, d_S^l)$$

where

$$d_s^l = \begin{cases} 1 & \text{if } s \in \mathcal{S}_l \\ 0 & \text{otherwise} \end{cases}$$

and $|\mathcal{S}_l|$ is the size of the set $\mathcal{S}_l$. The unit vectors d^l indicate the directions in which $\sum_{s \in \mathcal{S}_l} x_s$ are increasing. Now, if the directional derivative of $V(x)$, in every direction $d^l, l \in \mathcal{L}_-$, is negative for all $x \in \mathcal{X}_{\mathcal{L}_-}(b)$, we can deduce that $V(x)$ is decreasing on $\mathcal{X}_{\mathcal{L}_-}(b)$, which in turn proves inequality (13).

The directional derivative of $V(x)$ in the direction of d^l is

$$D_{d^l} V = \nabla V \cdot d^l$$

where ∇V stands for the gradient of V. We require

$$\nabla V \cdot d^l \leq 0, \ \forall l \in \mathcal{L}_-, \ x \in \mathcal{X}_{\mathcal{L}_-}(b)$$

Component $s \in \mathcal{S}$ of the gradient of V can be calculated as

$$(\nabla V)_s = U_s'(x_s) - \sum_{k \in \mathcal{L}_-^- \cap \mathcal{L}_s} \beta_k p_k'(b_k + \sum_{r \in \mathcal{S}_k} x_r - c_k) + \alpha \sum_{k \in \mathcal{L}_-^- \cap \mathcal{L}_s} \frac{\partial J^+}{\partial b_k} (1_{\mathcal{L}_-}(k)[b_k + \sum_{r \in \mathcal{S}_k} x_r - c_k])$$

Carrying out the dot product, we obtain the directional derivative $D_{d^l} V$ as

$$\frac{1}{\sqrt{|\mathcal{S}_l|}} \sum_{s \in \mathcal{S}_l} (U_s'(x_s) - \sum_{k \in \mathcal{L}_-^- \cap \mathcal{L}_s} \beta_k p_k'(b_k + \sum_{r \in \mathcal{S}_k} x_r - c_k) + \alpha \sum_{k \in \mathcal{L}_-^- \cap \mathcal{L}_s} \frac{\partial J^+}{\partial b_k} (1_{\mathcal{L}_-}(k)[b_k + \sum_{r \in \mathcal{S}_k} x_r - c_k]))$$

By Lemma 3.6, the last term of this expression is nonpositive. Thus, we can write

$$D_{d^l} V \leq \frac{1}{\sqrt{|\mathcal{S}_l|}} \sum_{s \in \mathcal{S}_l} (U_s'(x_s) - \sum_{k \in \mathcal{L}_-^- \cap \mathcal{L}_s} \beta_k p_k'(b_k + \sum_{r \in \mathcal{S}_k} x_r - c_k))$$

Since the utility functions U_s are strictly concave, U_s' is maximized at $x_s = m_s$ for each $s \in \mathcal{S}_l$. Similarly, due to the strict convexity of the penalty functions p_l, and the assumption that $p_l'(0) > 0$, p_l' is maximized at zero for each $l \in \mathcal{L}_-^-$. Using these observations, we can bound $D_{d^l} V$ from above by

$$D_{d^l} V \leq \frac{1}{\sqrt{|\mathcal{S}_l|}} \sum_{s \in \mathcal{S}_l} \left(U_s'(m_s) - \sum_{k \in \mathcal{L}_-^- \cap \mathcal{L}_s} \beta_k p_k'(0) \right)$$

Finally, we pick β_k's such that the following is satisfied for all subsets $\mathcal{L}_-^- \subseteq \mathcal{L}$

$$\sum_{s \in \mathcal{S}_l} \left(U_s'(m_s) - \sum_{k \in \mathcal{L}_-^- \cap \mathcal{L}_s} \beta_k p_k'(0) \right) \leq 0$$

which is always possible for β_k's large enough. One simple condition, for example, in the case when all β's are equal, is

$$\beta \geq \frac{U_s'(m_s)}{\sum_{k \in \mathcal{L}_-^- \cap \mathcal{L}_s} p_k'(0)}$$

for each source $s \in \mathcal{S}_l$. This condition holds if

$$\beta \geq \frac{U_s'(m_s)}{p_s'(0)}, \ \forall s \in \mathcal{S}$$

which is condition (11). □

IV. Implications of the Solution and Algorithms

Theorem 3.1 shows that for β satisfying (11), the optimal source rates x_s are determined by solving the parameterized static maximization problem (8)-(9):

$$\max_{x \in \mathcal{X}} \sum_{s \in \mathcal{S}} U_s(x_s)$$
$$\text{subject to} \quad \sum_{s \in \mathcal{S}_l} x_s \leq c_l - b_l, \ l = 1, \ldots, L$$

Note that the centralized solution of this maximization problem would drive the buffer lengths throughout the network to zero in one step from any initial condition, and choose the source rates so as to maximize the aggregate utility (subject to capacity constraints) from this point on. However, solving this problem centrally requires coordination among possibly all sources and is impractical for networks. For the case when $b_l \equiv 0$, $\forall l \in \mathcal{L}$, there are algorithms that achieve a decentralized solution. In [1], [2] such a solution is obtained by decomposing the objective of maximizing aggregate source utility into optimization subproblems for the network (i.e. links) and the sources. In [3], [4], the dual of the above optimization problem is used to come up with a decentralized algorithm that converges to the solution of (8)-(9) with no b_l's. In other words, the objective in [3] is to just maximize the aggregate source utility without taking into account the evolution of buffer lengths.

The basic algorithm given in [3] works as follows. At times $t = 1, 2, \ldots$, link l receives rates from all sources $s \in \mathcal{S}_l$ that go through it, and it calculates the aggregate rate $\sum_{s \in \mathcal{S}_l} x_s(t)$. With this information, it computes the so-called link price $p_l(t)$, which is updated according to

$$p_l(t+1) = [p_l(t) + \gamma(\sum_{s \in \mathcal{S}_l} x_s(t) - c_l)]_0^\infty \qquad (14)$$

This price is communicated to all sources $s \in S_l$ that use link l. Source s upon receiving the aggregate price $p^s(t) = \sum_{l \in \mathcal{L}_s} p_l(t)$ of all links in its path, chooses a new transmission rate $x_s(t)$ by solving the maximization problem

$$\max_{m_s \leq x_s \leq M_s} U_s(x_s) - x_s p^s \tag{15}$$

whose solution is trivially given by $x_s(t) = [U_s'^{-1}(p^s(t))]_{m_s}^{M_s}$.

In [3], it is shown that this distributed algorithm converges to the solution of (8)-(9) without the b_l's, if the curvatures of U_s are bounded away from zero, i.e. $-U_s''(x_s) \geq -1/\rho_s > 0$, $m_s \leq x_s \leq M_s$, and if the step size γ satisfies $0 < \gamma < 2/\rho \bar{L} \bar{S}$, where $\rho = \max_{s \in S} \rho_s$, $\bar{L} = \max_{s \in S} |\mathcal{L}_s|$, and $\bar{S} = \max_{l \in \mathcal{L}} |S_l|$.

Now if we apply the same algorithm as in [3] to our problem, the link price updates are modified to

$$p_l(t+1) = [p_l(t) + \gamma(b_l(t) + \sum_{s \in S_l} x_s(t) - c_l)]_0^\infty \tag{16}$$

It is yet to be shown that if the price (16) and queue length (1) updates along with the source law obtained from (15) will converge to the solution of the original maximization problem (8)-(9). Clearly, if we do have convergence, the equilibrium point of these update equations is the solution of (8)-(9).

Note that the price update (16) is very similar to REM [4], the convergence of which has not been established yet. (For the continuous time version of REM a proof of global stability is established in [12].) For discrete-time, it has been shown in [13] that REM converges for the single-link case. In [4], where REM was first introduced, it was motivated by the fact that the basic algorithm (14) fails to address the issue of large backlogs $b_l(t)$ when the step size γ is small. Buffer length $b_l(t)$ was introduced into the price update equation (16) to remedy this problem. Our analysis shows that REM is actually optimal in the sense that its limit maximizes the performance criterion (2).

V. Conclusions and Future Research

In this paper, we have described a dynamic optimization approach to flow control. We formulated the flow control problem as an infinite-horizon discounted optimal control problem, derived sufficient conditions for the existence of its solution, and obtained the solution under these conditions. The objective of the cost function is twofold: 1) maximize the aggregate source utility, 2) clear all buffers in the network.

The stationary policy that maximizes the infinite-horizon cost function is not decentralized in that coordination among possibly all sources is required to implement it on a network. However, as in the case of static optimization flow control, it is possible to come up with decentralized update schemes that converge to the optimal stationary policy. An immediate extension of this work would be to establish convergence of one of these algorithms. Also, instead of the infinite-horizon discounted cost function, one may look at maximizing the infinite-horizon average cost function, which may lead to a different policy of assigning rates to sources. However, a more interesting direction of research would be to see if there exists a finite or an infinite horizon cost function that would naturally lead to decentralized rate assignments for all sources in the network. Incorporating delay into the network model might be another extension of this work. In this case, the solution of the infinite-horizon problem might not be as simple to obtain. One may need to go back to the finite-horizon problem, and see how delay affects the information structure of the optimization problem. Finally, it may be interesting to investigate the stability properties of the algorithm under asynchronous updates as in [3].

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