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A Distributed Globally Convergent Algorithm for Fair, Queue-Length-Based Congestion Control

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Abstract

We present an easy-to-implement explicit rate congestion control algorithm for connection-oriented data networks. The algorithm is adaptive to changing network conditions, robust to network delays, and is decentralized. Further, it achieves max-min fairness and queue length stability under finite buffer length and minimum rate constraints. The proof of global convergence of the algorithm involves stability analysis of discrete-time linear systems with saturation.

1 Introduction

Performance of a data network is fundamentally linked to the ability of the network to provide quality-of-service (QoS) differentiation for different network applications. In contrast to the traditional, low-bandwidth, voice-based telephone network, ISDN (Integrated Service Digital Network) allows several types of network traffic (such as, voice and video) to coexist in the same transmission medium. The network traffic in ISDN networks can be broadly classified as guaranteed-service traffic and best-effort service traffic. Guaranteed service refers to a contract between the network service provider and the end user which requires the network to provide fixed quality of services (QoS) to the traffic. The QoS guarantees can be in the form of upper bounds on packet loss probability, delay, etc. In contracts, best-effort traffic is guaranteed very little *a priori*. In the context of ATM (Asynchronous Transfer Mode) networks, the best-effort traffic (in particular the Available Bit Rate (ABR) service) may be guaranteed a minimum rate (MR) and a bound on the loss rate (ratio of the number of lost packets to the total number of packets transmitted). Instead of guaranteeing fixed QoS parameters, the idea is to *fairly* allocate the network resources to competing users. In the Internet today, most users can be thought of generating best-effort type traffic, too. For instance, no traffic related service guarantees are made in advance when browsing a web page, or sending an email. Thus, in general, best-effort sources can adjust their rates to the level of available service, making it possible to control the congestion in the network.

In this paper, we consider the congestion control problem for best-effort type traffic in *connection oriented* data networks. In a connection-oriented network when a connection is established, a route from the source to the destination is chosen as part of the connection setup. This route is usually called a virtual circuit (VC), in analogy with the physical circuits set up by the telephone system. The same route or VC is used for all traffic flowing over the connection. When the connection is released, the virtual circuit is also terminated.

For best-effort sources to adapt their rates to changing network conditions, there must be a mechanism through which information about the state of the network is conveyed to the source. This information can be in the form of bandwidth availability, state of congestion, or impending congestion. In explicit-rate (ER) feedback congestion control, this is achieved by letting each source s periodically send out a special packet (termed as a resource management cell in the ATM context) with an ER field, which travels along the same route, $\mathcal{L}_s$, as the data packets but are treated specially by the switches along the way. This packet is eventually turned around by the destination and sent back to the source. The source initially sets the ER field to the rate it would like to transmit. As this special packet passes through various switches on the way to the destination and back to the source, those that are congested may reduce ER. When the source receives the ER field back, it adjusts its rate accordingly. As the links along the way may only reduce the ER field, the explicit rate received by the source will be the minimum of the rates dictated by the links along its path. As the link speeds continue to rise, the delay-bandwidth product (i.e., the product of the round-trip propagation delay and the link capacity) increases. An issue of importance that arises in this context is how to deal with action delays, as they may not be known accurately. In designing ER con-

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gestion control algorithms, another challenge is posed by the traffic characteristics of the network. The term connection lifetime refers to the duration of a VC. Some VCs may have a shorter connection lifetime than the smallest round-trip delay in the network. Controlling the rate of such VCs is hard, if not impossible, due to their short duration as compared to the time constant of the feedback loop.

In this paper, we introduce a control-based mathematical model that helps us address these design issues. Our initial modeling assumption is that of a single link shared by a number of best-effort sources. We assume that the switch that controls the link uses output queueing in which the packets destined to the same output line are queued on a finite-size output buffer, and are served on a FIFO (First In First Out) basis at the available rate. We propose a robust adaptive ER congestion control algorithm which requires only the knowledge of the maximum round-trip network delay and the maximum number of simultaneous connections. We assume that the switch keeps track of the queue length, available service rate, and aggregate rate of arriving packets at each one of its output lines, and show that if the controller gains are picked properly, our algorithm achieves max-min fairness along with queue length stability, despite inaccuracies in the knowledge of the actual number of best-effort sources and their corresponding round-trip delays. This analysis can be extended to a network with multiple links under some simplifying assumptions. The network level problem can be cast in the framework of hybrid systems. We have established a local stability result for the network problem, however this result is not included here due to space restrictions.

Performance Criteria

The main goal of any congestion control algorithm is to provide fairness among all VCs with a minimal loss rate and maximal utilization of network resources. The latter two of these objectives can be achieved by regulating the queue length at switches around a desirable level. Tracking such a nominal queue length (whose exact value is determined based on QoS requirements) is desirable in order to avoid losses due to overflow and waste of the buffer capacity due to underflow. The most widely accepted notion of fairness is the *max-min fairness* criterion [4]. Under this criterion, the fair share of a connection contending for a given link bandwidth should be equal to

$$\text{Fair share} = \mathrm{MR}_m + \frac{a - (u + \overline{\mathrm{MR}})}{N - M^u} \qquad (1)$$

Here MR_m is MR of connection m, $\overline{\mathrm{MR}}$ is the sum of MRs of active connections, a is the total available bandwidth on the given given link, N is the number of active connections, u is the sum of bandwidth of connections bottlenecked elsewhere (including those limited by their peak rates), and M^u is the number of such connections.

Previous Work

The simplest ER feedback control mechanism is called *rate matching*. In rate matching, the switch measures the average rate available to the sources at periodic intervals and simply divides a fraction of this capacity equally among the various users. This is the basic approach used in [1], although several modifications are used in the actual implementation. The main advantage of this scheme is its simplicity, but it is difficult to control queue length optimally to avoid buffer overflows. However, this scheme is stable (i.e., the queue length remains bounded in an appropriate stochastic sense). Queue length information is not used in the basic algorithm, although [1] allows one to incorporate queue length information in an ad hoc manner. Alternatively, this problem can be viewed as a feedback control problem where queue length is used for explicit feedback. This approach is used in [2] to study this problem using classical control techniques or using a state-space approach. As in rate matching, the primary goal is not optimality, but simply queue length stability. In these approaches, the available bandwidth is treated as an unmodeled disturbance. Thus, the algorithm in [2] ensure stability in the presence of this disturbance, but do not address the issue of performance, and require per-flow information. In a recently published work [3], a closed-loop proportional-derivative controller is proposed, which achieves max-min fairness plus queue length stability, but the design falls short of addressing the issue of robustness against uncertainty in delays.

Outline

The rest of the paper is organized as follows. In the next section, we describe the mathematical model of a basic switch. The algorithm is introduced in Section 3, and its analysis is carried out in Section 4. The paper ends with the concluding remarks of Section 5.

2 Basic Switch Model

We adopt a switch model that uses output queueing. In output queueing each output line of the switch has a finite buffer, where the incoming packets are served on a FIFO basis. Associated with each output line, there is an ER controller that suggests an explicit rate for the connections (VCs) using this line. Note that, from the standpoint of the network, each output line represents a link. In what follows we will focus on a particular link of an arbitrary switch in the network. The model we adopt here is a discrete-time model, where a time unit corresponds to the interval over which the rate available to sources is determined (that is, the interval over which measurements are made). Further, this measurement interval is assumed to be long enough for the switch to be able to process several packets —a reasonable assumption if the link speeds are high and packet sizes are small. This allows us to ignore the packet-level dynamics and model the traffic as a fluid.

Let $r(n)$ denote the total number of packets that arrive at the link buffer in the interval $[n, n+1)$, and let $a(n)$ denote the number of packets that depart from this buffer in the same time interval. Note that $a(n)$ represents the available bandwidth. Denoting the queue length at time n by $q(n)$ we have the evolution

$$q(n+1) = \min\left\{\overline{Q}, \max\left\{0, q(n) + r(n) - a(n)\right\}\right\} \quad (2)$$

where $\overline{Q}$ denotes the size of the output buffer. Let there be a total number of $N \geq 1$ connections (sources) switched through the output line under study, and the number of packets that arrive from source m during time-slot $[n, n+1)$ be denoted by $r_m(n)$. Clearly, $r(n) = \sum_{m=1}^{N} r_m(n)$. In general, $r(n)$ has two components: 1) The number of packets that arrive from *uncontrolled* sources, i.e., those sources which are bottlenecked elsewhere in the network or are limited by their peak rate constraints. The ER controller at this switch has no control over these sources. We denote this component of $r(n)$ by $u(n)$. 2) The number of packets that arrive from *controlled* sources, which are bottlenecked at this switch. In addition, we assume that each source, controlled or uncontrolled, has negotiated a nonnegative MR with the network. We designate the MR of the controlled source m by MR_m. Let there be a total number of $M \geq 1$ controlled sources, and the number of packets that arrive from controlled source m in the interval $[n, n+1)$ be denoted by $r_m^c(n)$. Then, we have the following relation between $r(n)$, $r^c(n)$, and $u(n)$: $r(n) = \sum_{m=1}^{M} r_m^c(n) + u(n)$. To have a feasible problem, we must guarantee that there is enough bandwidth for all sources to transmit at their minimum rates, which requires $a(n) - u(n) > \overline{\mathrm{MR}}$, where $\overline{\mathrm{MR}}$ denotes the sum of MRs of active connections. As mentioned in the introduction, it takes time from the moment the ER decision is made by the switch until an action is taken by a source, and until subsequently that action affects the state of the switch that initiated the action. Thus, the rate of source m at time n, $r_m(n)$, is actually an outcome of an action taken d_m time units earlier, where d_m represents the action delay for source m and is taken to be independent of time n. Thus, we have $r_m^c(n) = \mathrm{MR}_m + \mathrm{ER}(n - d_m)$, where $\mathrm{ER}(n)$ denotes the action of the switch (ER controller) at time n. We assume that $d_m \leq \overline{d}, \forall m$, where $\overline{d}$ corresponds to the maximum round-trip network delay. The action delay, d_m, for source m, consists of several components, such as the round-trip propagation delay, the queuing and processing delays, etc. Since queuing and processing delays are subject to variation, it is impossible for the switch to predict the exact value of d_m beforehand. As evident from (1), any calculation of the fair share at the switch requires the knowledge of the number of controlled connections, $M (= N - M^u)$. Due to the bursty nature of the best effort traffic, an accurate value of M may not be known either. Motivated by these observations, we want to develop an ER control algorithm which is robust to uncertainty in action delays, and at the same time adaptive to the number of controlled connections. We close our account on this section by rewriting the queue dynamics (2) in a more convenient form:

$$\begin{aligned} q(n+1) &= \min\{\overline{Q}, \max\{0, q(n) + \overline{\mathrm{MR}} + u(n) - a(n) \\ &\quad + \sum_{k=0}^{\overline{d}} m_k \mathrm{ER}(n-k)\}\} \end{aligned} \quad (3)$$

where m_k denotes the number of controlled sources having k units of action delay.

3 The Algorithm

Consider an arbitrary link in the network. Even though the available bandwidth and the aggregate rate of uncontrolled sources on this link may vary, to focus on the derivation of the algorithm, we assume that these quantities do not change with time. This assumption is justified if $a(n)$ and $u(n)$ vary slowly compared to the time constant of the closed-loop system. As mentioned in the preceding section, the number of controlled connections at time n, $M(n)$, may not be known to the switch. As $M(n)$ varies with time the fair share of the available bandwidth varies, too. We want the ER controller at the switch to track this variation in the number of controlled connections, and adapt its output to the new value of the fair share calculated by (1). In order to simplify the final design, we want to ease the effect of multiple time delays on the system dynamics. One way for the switch to do this is to wait long enough after issuing an explicit rate so that all controlled sources on the given link start to transmit at the issued rate. As the switch has an estimate of the round-trip delay of each connection on the link, putting an upper bound, $\overline{d}$, on the maximum round-trip delay is feasible. Thus, if the switch updates $\mathrm{ER}(n)$ every $(\overline{d}+1)$ time units, all controlled sources will have enough time to modify their transmission rates according to the explicit rate fed back at the previous update interval. For ease of notation, let us introduce the subsequences, $q_s(n) := q(n(\overline{d}+1))$, $\mathrm{ER}_s(n) := \mathrm{ER}(n(\overline{d}+1))$, and $r_s(n) := r(n(\overline{d}+1))$, which are obtained by downsampling the original sequences. Note that $\mathrm{ER}(n)$ is kept at the same value for an interval of length $(\overline{d}+1)$. To achieve the dual goal of max-min fairness and queue length stability, $\mathrm{ER}_s(n)$ will be updated according to

$$\begin{aligned}\mathrm{ER}_s(n) &= \min\{a, \max\{0, \mathrm{ER}_s(n-1) - \alpha(r_s(n) - a) \\ &\quad -\beta(q_s(n) - Q)\}\}, \quad \mathrm{ER}_s(0) = a \end{aligned} \tag{4}$$

where Q is the target queue length, and α and β are parameters to be selected to meet various design criteria. Here the max function is introduced to ensure that the switch asks the sources to transmit at a positive rate in excess of their minimum rates, as required by the QoS specifications, and the min function puts an upper bound on the maximum allowed rate for the sources. In (4), the term $-\beta(q_s(n) - Q)$ is introduced to drive the queue length to the desired set point by providing negative feedback in the closed-loop system dynamics. Note that, this algorithm does not require any per-flow information, as it only uses the aggregate rate $r_s(n)$. Now, if this algorithm stabilizes the system, $q_s(n)$ converges to Q, and $\mathrm{ER}_s(n)$ converges to $\mathrm{ER}_s(\infty) = \left[a - \left(u + \overline{\mathrm{MR}}\right)\right]/M$. Hence, if we can show the stability of the discrete-time system (3)-(4), the rate of controlled connection m r_m^c asymptotically achieves $r_m^c(\infty) = \mathrm{MR}_m + \left[a - \left(u + \overline{\mathrm{MR}}\right)\right]/M$, which is the minimum rate plus max-min fair share of the available bandwidth. In what follows, we will show that if the controller gains (α, β) are picked small enough, and if the buffer size, $\overline{Q}$, is large enough, with all these bounds given explicitly, then the algorithm converges globally to the desired equilibrium point.

4 Analysis of the Algorithm

4.1 State Space Formulation

We first introduce the shifted variables as in [5], $x(n) := q(n) - Q, x_s(n) := q_s(n) - Q$. Rewriting the queue and the controller dynamics in terms of the newly introduced variables, we obtain

$$\begin{aligned} x(n+1) &= \min\{\overline{Q} - Q, \max\{-Q, x(n) + \overline{\mathrm{MR}} + u - a \\ &\quad + \sum_{k=0}^{\overline{d}} m_k \mathrm{ER}(n-k)\}\} \end{aligned} \tag{5}$$

$$\begin{aligned} \mathrm{ER}_s(n) &= \min\{a, \max\{0, \mathrm{ER}_s(n-1) - \beta x_s(n) \\ &\quad -\alpha\left(M\mathrm{ER}_s(n-1) + \overline{\mathrm{MR}} + u - a\right)\}\} \end{aligned} \tag{6}$$

To find a recursive relation for the subsequence $x_s(n)$, we consider the evolution of $x(n)$ for $(\overline{d}+1)$ time units starting at time $n(\overline{d}+1)$ for an arbitrary $n \geq 0$. We have

$$\begin{aligned} x(n(\overline{d}+1)+1) &= \min\{\overline{Q} - Q, \max\{-Q, x(n(\overline{d}+1)) \\ &\quad +\overline{\mathrm{MR}} - a + u + m_0\mathrm{ER}_s(n) \\ &\quad +(M - m_0)\mathrm{ER}_s(n-1)\}\} \\ &\quad \vdots \\ x((n+1)(\overline{d}+1)) &= \min\{\overline{Q} - Q, \max\{-Q, \\ &\quad x((n+1)(\overline{d}+1) - 1) \\ &\quad +M\mathrm{ER}_s(n) + u + \overline{\mathrm{MR}} - a\}\}, \end{aligned} \tag{7}$$

where m_k's have the property that $0 \leq m_k \leq M, k = 1, \ldots, \overline{d}$, and furthermore $\sum_{k=0}^{\overline{d}} m_k = M$. Now, if we recursively substitute equations (7) into one another starting from the top one, we can express $x_s(n+1)$ in terms of $x_s(n)$, $\mathrm{ER}_s(n)$, and $\mathrm{ER}_s(n-1)$. Carrying out the manipulations, we obtain

$$\begin{aligned} x_s(n+1) &= \min\{\overline{Q} - Q, \max\{-Q, x_s(n) + \gamma_0\mathrm{ER}_s(n) \\ &\quad +\gamma_1\mathrm{ER}_s(n-1) + \frac{\gamma_0 + \gamma_1}{M}(u + \overline{\mathrm{MR}} - a)\}\} \end{aligned}$$

where

$$1 \leq \gamma_0 \leq \overline{\gamma}_0 \quad := \sum_{k=0}^{\overline{d}} (\overline{d} + 1 - k) m_k \tag{8}$$

$$0 \leq \gamma_1 \leq \overline{\gamma}_1 \quad := \sum_{k=0}^{\overline{d}} k m_k. \tag{9}$$

where, in general $\gamma_0 + \gamma_1 = kM$, for some $1 \leq k \leq (\overline{d}+1)$. Note that $\overline{\gamma}_0$ and $\overline{\gamma}_1$ satisfy $\overline{\gamma}_0 + \overline{\gamma}_1 = (\overline{d}+1)M$. Let $e := u + \overline{\mathrm{MR}}$. The original system (5)-(6) can be written in terms of $x_s(n)$ as follows:

$$\begin{aligned} x_s(n+1) &= \min\{\overline{Q} - Q, \max\{-Q, x_s(n) + \gamma_0\mathrm{ER}_s(n) \\ &\quad +\gamma_1\mathrm{ER}_s(n-1) + \frac{\gamma_0 + \gamma_1}{M}(e - a)\}\} \end{aligned} \tag{10}$$

$$\begin{aligned} \mathrm{ER}_s(n) &= \min\{a, \max\{0, (1 - \alpha M)\mathrm{ER}_s(n-1) \\ &\quad -\alpha(e - a) - \beta x_s(n)\}\} \end{aligned} \tag{11}$$

Before proceeding any further, we want to show that the saturation nonlinearities imposed on $\mathrm{ER}_s(n)$ are redundant provided that α and β are picked properly. In other words, we claim that if $0 \leq \mathrm{ER}_s(0) \leq a$, which is the case on account of (4), then $0 \leq \mathrm{ER}_s(n) \leq a$ for all $n \geq 1$, which in turn implies that the saturation nonlinearities in (11) never become active. Let $y(n)$ denote the linear version of the right-hand side of (11), i.e. $y(n) := (1 - \alpha M)y(n-1) - \alpha(e - a) - \beta x_s(n)$, and assume that

$$0 \leq \alpha \leq \frac{1}{M}, \quad 0 \leq \beta \leq \min\{\frac{\alpha(a - e)}{\overline{Q} - Q}, \frac{\alpha[Ma + e]}{Q}\} \tag{12}$$

Fix $k_0 \geq 0$, and let $0 \leq y(k_0) \leq a$. Then we can show that $0 \leq y(k_0 + 1) \leq a$. Since k_0 is arbitrary, we have $0 \leq y(n) \leq a$ for all $n \geq 1$, provided that $0 \leq y(0) \leq a$, and the pair (α, β) is picked in accordance with (12). Consequently, we can remove the min and max functions from (11), and substitute for $\mathrm{ER}_s(n)$ in (10). This results in

$$x_s(n+1) = \min\{\overline{Q}-Q, \max\{-Q, (1-\beta\gamma_0)x_s(n) + [\gamma_0(1-\alpha M)+\gamma_1]\mathrm{ER}_s(n-1) - \gamma_0\alpha(e-a) + \frac{\gamma_0+\gamma_1}{M}(e-a)\}\}$$

Now, we are in a position to write the system dynamics in the state-space from. Introducing the state variables $x_0(n) := x_s(n), x_1(n) := \mathrm{ER}_s(n-1) - \frac{a-e}{M}$, we obtain

$$x_0(n+1) = \min\{\overline{Q}-Q, \max\{-Q, (1-\beta\gamma_0)x_0(n) + [\gamma_0(1-\alpha M)+\gamma_1]x_1(n)\}\} \quad (13)$$

$$x_1(n+1) = \min\left\{a-\frac{a-e}{M}, \max\left\{-\frac{a-e}{M}, -\beta x_0(n) + (1-\alpha M)x_1(n)\right\}\right\} \quad (14)$$

which we can write as $x(n+1) = \mathrm{sat}(Ax(n))$, for $x = [x_0\ x_1]^T \in \mathcal{R}^2$, and

$$A = \begin{bmatrix} 1-\beta\gamma_0 & \gamma_0(1-\alpha M)+\gamma_1 \\ -\beta & 1-\alpha M \end{bmatrix} \quad (15)$$

Let $y = [y_0\ y_1]^T$. We define the saturation function as $\mathrm{sat}(y) := [\mathrm{sat}_0(y_0)\ \mathrm{sat}_1(y_1)] : \mathcal{R}^2 \to \mathcal{R}^2$, where

$$\mathrm{sat}_0(y_0) := \begin{cases} \overline{Q}-Q, & y_0 > \overline{Q}-Q \\ y_0, & -Q \le y_0 \le \overline{Q}-Q \\ -Q, & y_0 < -Q \end{cases}$$

$$\mathrm{sat}_1(y_1) := \begin{cases} a-\frac{a-e}{M}, & y_1 > a-\frac{a-e}{M} \\ y_1, & -\frac{a-e}{M} \le y_1 \le a-\frac{a-e}{M} \\ -\frac{a-e}{M}, & y_1 < -\frac{a-e}{M} \end{cases}$$

Note that, even though $x_1(n)$ never saturates, we have included the saturation nonlinearities in (14), since this simplifies the analysis of the algorithm.

4.2 A Robust Global Stability Result

In this section, we will show that the class of nonlinear systems (13)-(14) with an arbitrary pair of (γ_0, γ_1) values satisfying (8)-(9), is globally asymptotically stable if the controller gains (α, β) are picked appropriately. This result will enable us to conclude that our ER congestion control algorithm achieves max-min fairness along with queue length stability under minimum rate and finite buffer length constraints.

Lemma 5.1: *The system (13)-(14) has a unique equilibrium point at the origin.*

Now, let $\mathcal{D}$ denote the region of the state-space for which the saturation nonlinearities are not active. We will say that the system (13)-(14) is *stable* if $x_e = 0$ is globally asymptotically stable. It is clear that for any $x(0) \notin \mathcal{D}, x(n) \in \mathcal{D}, n \ge 1$, will always be true. Thus, without any loss of generality, we will assume that $x(0) \in \mathcal{D}$. It can be shown that the linear system obtained by removing the saturation nonlinearities is stable.

Lemma 5.2: *The linear system corresponding to (13)-(14)*

$$x^l(n+1) = Ax^l(n) \quad (16)$$

is globally asymptotically stable, if α and β satisfy

$$0 < \alpha < \frac{1}{M}, \quad 0 < \beta < \min\left\{\frac{1}{\gamma_0}, \frac{\alpha M}{\gamma_1}\right\} \quad (17)$$

In order to prove global asymptotic stability, we use a result from digital filter design with overflow arithmetic. In [6] a set of sufficient conditions is given for the stability of a nonlinear system $x(n+1) = \mathrm{sat}(Ax(n))$, for $A \in \mathcal{R}^{2\times 2}$, where $\mathrm{sat}(x)$ is of the form

$$\mathrm{sat}_i(x_i) = \begin{cases} 1, & x_i > 1 \\ x_i, & |x_i| \le 1 \\ -1, & x_i < -1 \end{cases}$$

In particular, the nonlinear system $x(n+1) = \mathrm{sat}(Ax(n))$ is asymptotically stable if $A = [a_{ij}]$ is stable, and in addition the following condition is met:

$$|a_{11} - a_{22}| \le 2\min\{|a_{12}|, |a_{21}|\} + 1 - \det(A)$$

where $\det(A)$ denotes the determinant of A. For ease of notation, we define $\Lambda := \max\{|a_{12}|, |a_{21}|\}$, $\lambda := \min\{|a_{12}|, |a_{21}|\}$, $Q_{\max} := \max\{\overline{Q}-Q, Q\}$, $Q_{\min} := \min\{\overline{Q}-Q, Q\}$, $A_{\max} := \max\{a-(a-e)/M, (a-e)/M\}$, and $A_{\min} := \min\{a-(a-e)/M, (a-e)/M\}$. The result of [6] cannot be directly applied to our system (since the saturation bounds in our case are not symmetric), but following the same procedure as in [6], we can prove the following result.

Lemma 5.3: *The nonlinear system (13)-(14) is globally asymptotically stable, if $A = [a_{ij}]$ is stable, and the following conditions are satisfied:*

$$|a_{11} - a_{22}| \le 2\lambda + 1 - \det(A) \quad (18)$$

$$Q_{\max} > p_{\max}A_{\min}, \quad Q_{\min}Q_{\max} > p_{\max}A_{\min}A_{\max} \quad (19)$$

where $p_{\max}$ is given by

$$p_{\max} = \frac{1-\beta\gamma_0}{1-\alpha M} + \frac{(\alpha M - \beta\gamma_1)^2}{2(1-\alpha M)^2}\left(1 + \frac{1}{\sqrt{1 - \frac{(\alpha M - \beta\gamma_0)^2}{(\alpha M - \beta\gamma_1)^2 + 4(1-\beta\gamma_0)(1-\alpha M)}}}\right)$$

Proof: We define $V(x) := x^T P x$, where $P = [p_{ij}], i,j = 1,2$, is a 2×2 positive definite matrix, and hence

$$p_{11} > 0,\ p_{22} > 0,\ p_{11}p_{22} > (p_{12})^2 \quad (20)$$

$V(x)$ will serve as a Lyapunov function for the system (13)-(14). First, we require that

$$P - A^T P A \geq 0 \tag{21}$$

for the stability of the corresponding linear system $x^l(n+1) = Ax^l(n)$. Note that if A is stable there always exists a positive definite P such that (21) is satisfied, but we want to characterize the family of P matrices satisfying (21). Now, if we assume that A is stable, the condition (21) can actually be reduced to

$$\det\left(P - A^T P A\right) \geq 0 \tag{22}$$

Next, for any $x \in \mathcal{R}^2$, $x \notin \mathcal{D}$ we require $V(\text{sat}(x)) < V(x)$. It can be shown that this condition is satisfied for the system (13)-(14), if

$$|p_{12}| \leq \min\left\{p_{11}\frac{Q_{\min}}{A_{\max}}, p_{22}\frac{A_{\min}}{Q_{\max}}\right\} \tag{23}$$

Observe that under (17), we have $a_{11} > 0$, $a_{22} > 0$, and $a_{12}a_{21} < 0$. Now, if $a_{12}a_{21} < 0$, the condition $\det\left(P - A^T P A\right) \geq 0$ is equivalent to

$$\begin{aligned} &p_2|a_{11} - a_{22}|^2 - 2p_1(\Lambda + p_2\lambda)|a_{11} - a_{22}| + 4p_1^2\lambda\Lambda \\ &+(\Lambda - p_2\lambda)^2 \leq \left(p_2 - p_1^2\right)\left(1 - \det(A)\right)^2 \end{aligned} \tag{24}$$

where we let $p_{11} = 1$, $p_2 = p_{22}$, and $p_1 = |p_{12}|$. This inequality represents an ellipse (including its interior) in the (p_1, p_2) plane. In terms of the newly introduced variables, condition (23) becomes

$$p_1 \leq \min\left\{\frac{Q_{\min}}{A_{\max}}, p_2\frac{A_{\min}}{Q_{\max}}\right\} \tag{25}$$

Thus, the matrix P satisfies (20) and (22) if (p_1, p_2) is a point on the face of an ellipse, described by (24), with $p_2 > p_1^2$. Then, we can show that condition (25) is satisfied together with (24), and $p_2 > p_1^2$ for at least one point in the (p_1, p_2) plane if the matrix A is constrained by (18), and $Q_{\min}$ and $Q_{\max}$ are constrained by (19). Details are omitted due to space limitations. Next, we state the main result of this section.

Theorem 5.1: *The class of nonlinear systems (13)-(14) with an arbitrary pair of (γ_0, γ_1) values satisfying (8)-(9) is globally asymptotically stable, if (α, β) satisfy*

$$0 < \alpha < \frac{2}{3}\frac{1}{M}, \quad 0 < \beta < \min\left\{\frac{1}{\overline{\gamma_0}}, \frac{\alpha M}{\overline{\gamma_1}}\right\}$$

and $\overline{Q}$ and Q satisfy (19) with $p_{\max} \geq 5 + \frac{4}{\sqrt{3}}$.

Remark 5.1: Controller gains can in fact be chosen to be independent of M and (γ_0, γ_1), as follows:$0 < \alpha < \frac{2}{3}\frac{1}{\overline{M}} := \overline{\alpha}$, $0 < \beta < \frac{\alpha}{\overline{d}+1} := \overline{\beta}$ The following Corollary relates the main result back to the original problem.

Corollary 5.1: *If $\left(\alpha, \beta, \overline{Q}, Q\right)$ are selected as*

$$0 < \alpha < \tfrac{2}{3}\tfrac{1}{\overline{M}}, \quad Q_{\max} > p_{\max}A_{\min}$$
$$Q_{\min}Q_{\max} > p_{\max}A_{\min}A_{\max}$$
$$0 < \beta < \alpha\min\left\{\tfrac{1}{\overline{d}+1}, \tfrac{\alpha(a-e)}{\overline{Q}-Q}, \tfrac{\alpha[(M-1)a+e]}{Q}\right\}$$

then the ER *control algorithm (4) asymptotically achieves*

$$\lim_{n\to\infty}\{q(n), \text{ER}(n)\} = \left\{Q, \frac{a - \left(u + \overline{\text{MCR}}\right)}{M}\right\}$$

resulting in a max-min fair bandwidth allocation along with a stable queue length.

5 Conclusions

In this paper, we have presented a decentralized and robust ER congestion control algorithm, and have shown that it performs well under various criteria, such as max-min fairness, and MR constraints. The convergence proof of this algorithm involves stability analysis of linear systems with saturation type nonlinearities. An extensive ns2 simulation study of the ER algorithm is being conducted to see how well it reacts to network latencies, as well as to evaluate its performance under various scenarios.

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