Nonnegative State-Space Realizations of LTI Systems

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NONNEGATIVE STATE-SPACE MODELS

DT State-Space Systems

$$\begin{aligned} x[k+1] &= Ax[k] + Bu[k] \\ y[k] &= Cx[k] + Du[k] \end{aligned}$$

CT State-Space Systems

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

Two equivalent definitions:

- Nonnegative initial state and inputs generate nonnegative state variables and outputs
- All matrices (A,B,C, and D) have nonnegative entries (diagonal of A can have negative entries in CT)

Motivation:

- Modeling systems with variables constrained to be nonnegative
- Applications in economics, compartmental systems, biology, etc.

PROBLEM DESCRIPTION

Nonnegative Realization: Characterize all causal responses ($h[k]$ or $h(t)$) that have nonnegative realization

Minimality: Least number of variables in nonnegative realization

Outline

- Background
- Role of maximum modulus poles (DT case)
- Criterion for systems with multiple inputs and outputs
- Extension to continuous-time systems
- Minimality, bounds on size of minimal realizations

CONE DEFINITIONS

Definition 1: The set $\mathcal{P} \subset R^n$ is called a *convex cone* if for any two n-tuples $p_i, p_j \in \mathcal{P}$,

$$\alpha p_i + \beta p_j \in \mathcal{P} \quad \text{for} \quad \alpha \geq 0 \;, \quad \beta \geq 0$$

Definition 2: Cone $\mathcal{P} \subset R^n$ is *polyhedral* if it is generated by a finite number of n-tuples $\{p_1, p_2, ..., p_N\}$.

We write $\mathcal{P} = cone(P)$, where $P = [\ p_1 \mid p_2 \mid \ \ldots \ \mid p_N\ ]$.

Definition 3: The dual of cone $\mathcal{P} \subset R^n$ is denoted by $\mathcal{P}^*$ and is given by

$$\mathcal{P}^* = \{\ y \in R^n \mid p^T y \geq 0 \text{ for } p \in \mathcal{P}\}$$

CRITERION FOR NONNEGATIVE REALIZABILITY (KODAMA ET AL.)

$$H(z) = \frac{p_1 z^{n-1} + p_2 z^{n-2} + ... + p_n}{z^n + q_1 z^{n-1} + ... + q_n} = \sum_{k=1}^{+\infty} h[k] z^{-k}$$

Minimal state-space realization (A_c, b_c, c_c^T) with n state variables

Definitions:

- $\mathcal{R} = \mathit{cone}\left(\{b_c, A_c b_c, A_c^2 b_c, ...\}\right)$
- $\mathcal{O} = \{x \mid c_c^T A_c^i x \geq 0 \text{ for } i = 0, 1, 2, ...\}$

Theorem: $H(z)$ has nonnegative realization iff there exists a *polyhedral* cone $\mathcal{P}$ such that:

(i) $A_c \mathcal{P} \subset \mathcal{P}$

(ii) $\mathcal{R} \subset \mathcal{P} \subset \mathcal{O}$

DE-MYSTIFYING KODAMA'S THEOREM

Theorem (Kodama et al.): $H(z)$ has nonnegative realization iff there exists polyhedral cone $\mathcal{P}$ such that: (i) $A_c\mathcal{P} \subset \mathcal{P}$, and (ii) $\mathcal{R} \subset \mathcal{P} \subset \mathcal{O}$.

Re-statement: $H(z)$ has nonnegative realization iff there exists $n \times N$ $(N \geq n)$ matrix P such that

$$A_cP = PA \; , \quad b_c = Pb \; , \quad c^T = c_c^T P$$

and A, b, and c^T have *nonnegative* entries. (A, b, c^T) realizes $h[k]$

$$h[k] = c_c^T A_c^{k-1} b_c = c^T A^{k-1} b$$

Reasoning:

- Polyhedral $\mathcal{P} \Leftrightarrow$ Matrix $P = [\, p_1 \mid p_2 \mid \cdots \mid p_N \,]$ (size $n \times N$), $\mathcal{P} = \mathit{cone}(P)$
- $A_c\mathcal{P} \subset \mathcal{P}$ and polyhedral $\mathcal{P}$ implies $A_cP = PA$ for a *nonnegative* matrix A
- $\mathcal{R} \subset \mathcal{P}$ implies $b_c = Pb$ for a *nonnegative* vector b
- $\mathcal{P} \subset \mathcal{O}$ implies that $c_c^T P$ is *nonnegative* (call it c)

CONDITIONS ON MAXIMUM MODULUS POLE(S) (ANDERSON ET AL.)

$$H(z) = \frac{p_1 z^{n-1} + p_2 z^{n-2} + \ldots + p_n}{z^n + q_1 z^{n-1} + \ldots + q_n} = \sum_{k=1}^{+\infty} h[k] z^{-k} \; , \qquad h[k] \geq 0$$

Conditions for nonnegative realizability

Necessary Condition	Sufficient Condition
$H(z)$ has a real, nonnegative pole of maximum modulus	$H(z)$ has a *unique* pole of maximum modulus (real and nonnegative). Multiple poles at that location are also allowed.
The poles of maximum modulus are those allowed for eigenvalues of maximum modulus in nonnegative matrices	$H(z)$ has multiple poles of maximum modulus $\bar{\lambda}$ (one of them real and nonnegative) *and* $\lim_{k \longmapsto \infty} \inf \bar{\lambda}^{-k} h[k] > 0$

RESULTS ON NONNEGATIVE MATRICES

Theorem (Perron-Frobenius): A nonnegative matrix A has a *real* nonnegative eigenvalue $\lambda_0 \geq 0$, such that $\lambda_0 \geq |\lambda|$ where λ is *any* eigenvalue of A. The corresponding eigenvector v_0 is also real and nonnegative.

More generally, the eigenvalues of maximum modulus $\bar{\lambda} \geq 0$ in an $N \times N$ nonnegative *irreducible* matrix are of the form

$$\{\bar{\lambda}\omega_1, \bar{\lambda}\omega_2, \ldots, \bar{\lambda}\omega_k\}$$

where $k \in \{1, 2, \ldots, N\}$ and $\omega_i^k = 1$ (i.e., ω_i are the kth roots of unity).

Note: *Reducible* A means that $PAP^T = \left[\begin{array}{cc} \bar{A}_{11} & 0 \\ \bar{A}_{21} & \bar{A}_{22} \end{array}\right]$ for some permutation P ($\bar{A}_{11}$ and $\bar{A}_{22}$ are square matrices).

NECESSARY CONDITION ON MAXIMUM MODULUS POLE(S)

Nonnegative (A, b, c^T) realizes $H(z)$

Key Observation (Anderson et al.): If the real and nonnegative eigenvalue of A of maximum modulus is *not* a pole of $H(z)$, then we can *rearrange* the state variables in (A, b, c^T) so that:

$$A = \begin{bmatrix} \bar{A} & 0 \\ * & * \end{bmatrix}, b = \begin{bmatrix} \bar{b} \\ * \end{bmatrix}, c^T = \begin{bmatrix} \bar{c} & 0 \end{bmatrix}, \text{ or}$$

$$A = \begin{bmatrix} \bar{A} & * \\ 0 & * \end{bmatrix}, b = \begin{bmatrix} \bar{b} \\ 0 \end{bmatrix}, c^T = \begin{bmatrix} \bar{c} & * \end{bmatrix}$$

Nonnegative $(\bar{A}, \bar{b}, \bar{c^T})$ also realizes $H(z)$

Conclusion: Poles of maximum modulus of $H(z)$ need to include a nonnegative real pole and be a *subset* of the allowable eigenvalues of maximum modulus in a nonnegative matrix.

NECESSARY CONDITION ON MAXIMUM MODULUS POLE(S)

Example: Consider $h[k] = \sin^2(2k)$ for $k \geq 1$ ($h[k] = 0$, otherwise)

$$H(z) = \frac{\frac{1}{2}}{1 - z^{-1}} - \frac{\frac{1}{2}(1 - \cos 4z^{-1})}{1 - 2\cos 4z^{-1} + z^{-2}} = \frac{\frac{1}{2}(1 - \cos 4)z^{-1}(1 + z^{-2})}{(1 - z^{-1})(1 - 2\cos 4z^{-1} + z^{-2})}$$

Three poles of maximum modulus at 1, e^{j4}, e^{-j4}.

<u>Conclusion:</u> $h[k]$ does *not* have a nonnegative realization.

<u>Recall:</u> Eigenvalues of maximum modulus of $N \times N$ *irreducible* nonnegative matrix are of the form

$$\bar{\lambda} e^{\frac{2\pi j}{k}} \quad \text{for } j = 0, 1, ..., (k-1) \ ,$$

where $\bar{\lambda}$ is the maximum modulus and $k \in \{1, 2, 3, ..., N\}$.

CONDITIONS ON MAXIMUM MODULUS POLE(S) (ANDERSON ET AL.)

$$H(z) = \frac{p_1 z^{n-1} + p_2 z^{n-2} + \dots + p_n}{z^n + q_1 z^{n-1} + \dots + q_n} = \sum_{k=1}^{+\infty} h[k] z^{-k} \ , \qquad h[k] \geq 0$$

Conditions for nonnegative realizability

Necessary Condition	Sufficient Condition
$H(z)$ has a real, nonnegative pole of maximum modulus	$H(z)$ has a *unique* pole of maximum modulus (real and nonnegative). Multiple poles at that location are also allowed.
The poles of maximum modulus are those allowed for eigenvalues of maximum modulus in nonnegative matrices	$H(z)$ has multiple poles of maximum modulus $\bar{\lambda}$ (one of them real and nonnegative) *and* $\lim_{k \longmapsto \infty} \inf \bar{\lambda}^{-k} h[k] > 0$

SUFFICIENT CONDITION ON MAXIMUM MODULUS POLE(S)

Example: Consider $h[k] = 2^k + \sin^2(2k)$ for $k \geq 1$ ($h[k] = 0$, otherwise)

$$H(z) = \frac{2z^{-1}}{1-2z^{-1}} + \frac{\frac{1}{2}}{1-z^{-1}} - \frac{\frac{1}{2}(1-\cos 4z^{-1})}{1-2\cos 4z^{-1}+z^{-2}}$$

- One pole of maximum modulus (at 2)
- Nonnegative realization exists

Example: Consider $h[k] = 2^k - (-2)^k + \sin^2 k$ for $k \geq 1$ ($h[k] = 0$, otherwise)

$$H(z) = \frac{2z^{-1}}{1-2z^{-1}} + \frac{2z^{-1}}{1+2z^{-1}} + \frac{\frac{1}{2}}{1-z^{-1}} - \frac{\frac{1}{2}(1-\cos 2z^{-1})}{1-2\cos 2z^{-1}+z^{-2}}$$

- Two poles of maximum modulus (at 2, -2)
- $\lim_{k \mapsto \infty} \inf |2|^{-k} h[k] = 0$
- Theorem is inconclusive

SUFFICIENT CONDITIONS ON MAXIMUM MODULUS POLE(S)

Example: $h[k] = 2^k - (-2)^k + \sin^2 k$ for $k \geq 1$ ($h[k] = 0$, otherwise)

- Assume $h[k] = c^T A^{k-1} b$ for nonnegative (A, b, c^T)
- $h'[k] \equiv h[2k]$ satisfies $h'[k] = h[2k] = c^T A^{2k-1} b = c^T (A^2)^{k-1} b$ with nonnegative realization (A^2, Ab, c^T)
- However, $h'[k] = h[2k] = \sin^2 2k$ does *not* have a nonnegative realization
- Contradiction

Conclusion: $h[k]$ *cannot* have a nonnegative realization

MIMO Nonnegative Realizations

Notation:

$$H[k] = \begin{bmatrix} h_{1,1}[k] & h_{1,2}[k] & \dots & h_{1,m}[k] \\ h_{2,1}[k] & h_{2,2}[k] & \dots & h_{2,m}[k] \\ \vdots & \vdots & \ddots & \vdots \\ h_{p,1}[k] & h_{p,2}[k] & \dots & h_{p,m}[k] \end{bmatrix} \qquad H = \begin{bmatrix} H[1] \\ H[2] \\ H[3] \\ \vdots \end{bmatrix}$$

Nonnegative realization (A, B, C, D) for $H[k]$ means:

$$H[k] = \begin{cases} 0 & k < 0 \\ D & k = 0 \\ CA^{k-1}B & k \geq 1 \end{cases}$$

Definition: A backward-k-shift-invariant cone $\mathcal{X} \subset R^{\infty}$ satisfies

$$x \equiv \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{bmatrix} \in \mathcal{X} \;\Rightarrow\; \sigma^k(x) \equiv \begin{bmatrix} x_{k+1} \\ x_{k+2} \\ x_{k+3} \\ \vdots \end{bmatrix} \in \mathcal{X}$$

CRITERION FOR MIMO NONNEGATIVE REALIZATION (VAN DEN HOF)

For nonnegative (A, B, C, D) consider the matrix $X = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix} \subset R_+^{\infty \times n}$

Observations:

- $\mathcal{X} = cone(X)$ is polyhedral
- $H = XB \Rightarrow cone(H) \subset \mathcal{X}$
- $\mathcal{X}$ is backward-p-shift-invariant (since $\sigma^p(X) = XA$)

MIMO Criterion (van den Hof): $H[k]$ has nonnegative realization iff there exists a *polyhedral* $\mathcal{X} = cone(X) \subset R_+^{\infty}$ such that:

(i) $cone(H) \subset \mathcal{X}$, and

(ii) $\mathcal{X}$ is backward-p-shift-invariant

Extensions to Nonnegative Realizations in Continuous-Time

Observations:

- If (A, b, c^T) realizes $h(t)$, $((A + \alpha I), b, c^T)$ realizes $e^{\alpha t} h(t)$
- If (A, b, c^T) is a nonnegative realization in continuous-time, then for $\alpha \geq \alpha_0$:
  - *All* realizations $((A + \alpha I), b, c^T)$ have nonnegative matrices
  - $((A + \alpha I), b, c^T)$ corresponds to a nonnegative *discrete-time* realization

Approach:

CT Realization

DT Realization

$$\begin{tikzcd}
h(t) \text{ realized by } (A_c, b_c, c_c^T) \arrow[r, "+\alpha I"] & ((A_c + \alpha I), b_c, c_c^T) \text{ realizes } h[k] \text{ (or } e^{\alpha t} h(t)) \arrow[d] \\
h(t) \text{ realized by } ((A - \alpha I), b, c^T) & (A, b, c) \text{ realizes } h[k] \text{ (or } e^{\alpha t} h(t)) \arrow[l, "-\alpha I"']
\end{tikzcd}$$

CT TO DT REDUCTION

Example: $h(t) = e^{-t} - \frac{1}{2}e^{-\frac{5}{2}t}$ (for $t \geq 0$) has realization

$$A_{ct} = \begin{bmatrix} -1 & 0 \\ 0 & -2.5 \end{bmatrix}, \; b_{ct} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; c_{ct}^T = \begin{bmatrix} 1 & -0.5 \end{bmatrix}$$

For $\alpha = 2$, realization $((A_{ct} + 2I), b_{ct}, c_{ct}^T)$ is given by

$$A_{dt} = \begin{bmatrix} 1 & 0 \\ 0 & -0.5 \end{bmatrix}, \; b_{dt} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; c_{dt}^T = \begin{bmatrix} 1 & -0.5 \end{bmatrix}$$

A nonnegative realization of the response $h_{dt}[k] = c_{dt}^T A_{dt}^{k-1} b_{dt}$ is

$$A = \begin{bmatrix} 0 & 0.5 \\ 1 & 0.5 \end{bmatrix}, \; b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; c^T = \begin{bmatrix} 2 & 0.5 \end{bmatrix}$$

Result: Nonnegative realization for $h(t)$ is given by

$$((A - 2I), b, c^T) = \left(\begin{bmatrix} -2 & 0.5 \\ 1 & -1.5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 & 0.5 \end{bmatrix} \right)$$

CRITERIA FOR CONTINUOUS-TIME NONNEGATIVE REALIZATION (1)

$$H(s) = \frac{p_1 s^{n-1} + p_2 s^{n-2} + ... + p_n}{s^n + q_1 s^{n-1} + ... + q_n} \; , \qquad h(t) \geq 0 \;\text{ for } t \geq 0$$

General Realization (A_c, b_c, c_c^T)

Theorem (Anderson et al.): $h(t)$ has a nonnegative realization iff:

(i) There exists α such that the *discrete-time* transfer function

$$H(z) = c_c^T \left(zI - (A_c + \alpha I)\right)^{-1} b_c$$

(with realization $((A_c + \alpha I), b, c^T)$) has a discrete-time nonnegative realization, and

(ii) There is a *unique* (possibly multiple) *real* pole of $H(s)$ with maximal real part

Criteria for Continuous-Time Nonnegative Realization (2)

$$(A_c, B_c, C_c, D_c) \quad \text{realizes } H(t) \geq 0 \ \text{ for } t \geq 0$$

CT Theorem (van den Hof): $H(t)$ has nonnegative realization iff there exists α, such that the discrete-time response with realization $((A_c + \alpha I), B_c, C_c, D_c)$ has nonnegative realization (A, B, C, D).

Final Result: $H(t)$ has nonnegative realization $((A - \alpha I), B, C, D)$.

Minimality Using Nonnegative Rank (van den Hof)

Definition 1: The nonnegative rank of a nonnegative $p \times q$ matrix H is the least integer n for which there exists a matrix factorization $H = H_1 H_2$, where H_1 is a $p \times n$ *nonnegative* matrix and H_2 is an $n \times q$ *nonnegative* matrix.

Definition 2: Hankel matrix of n-dimensional realization (A, B, C, D)

$$\mathcal{H}(r,s) \equiv \begin{bmatrix} CB & CAB & \dots & CA^{s-1}B \\ CAB & CA^2B & & \vdots \\ \vdots & & & \vdots \\ CA^{r-1}B & \dots & \dots & CA^{r+s-2}B \end{bmatrix} = \begin{bmatrix} H[1] & H[2] & \dots & H[s] \\ H[2] & H[3] & & \vdots \\ \vdots & & & \vdots \\ H[r] & \dots & \dots & H[r+s-1] \end{bmatrix}$$

Fact: Nonnegative rank of $\mathcal{H}(r,s)$ smaller or equal to n for all $r, s \in Z_+$, since

$$\mathcal{H}(r,s) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{r-1} \end{bmatrix} \begin{bmatrix} B & AB & \dots & A^{s-1}B \end{bmatrix}$$

LOWER BOUND ON MINIMAL REALIZATIONS (1)

Example: Nonnegative impulse response $h[k]$ (for $k \geq 1$)

$$h[k] = 1 - \frac{1}{3}\left(-\frac{9}{10}\right)^{k-1} + \frac{1}{3}\left(-\frac{8}{9}\right)^{k-1} - \frac{1}{3}\left(-\frac{7}{8}\right)^{k-1}$$

has poles at 1, $-\frac{9}{10}$, $-\frac{8}{9}$, and $-\frac{7}{8}$.

- Minimal canonical realization has order 4
- Nonnegative realization (A, b, c^T) of order 4 is not possible:

$$trace(A) = 1 + -\frac{9}{10} + -\frac{8}{9} + -\frac{7}{8} \cong -1.67 < 0$$

- Contradiction; $trace(A) \geq 0$ since A is nonnegative

Conclusion: A nonnegative realization for $h[k]$ has more than 4 state variables.

How about a nonnegative realization with 5 state variables ?

LOWER BOUND ON MINIMAL REALIZATIONS (2)

Causal nonnegative response $h[k]$ for $k \geq 1$ ($h[k] = 0$, otherwise)

(i) Strictly proper *reduced* rational transfer function $H(z)$

(ii) Poles at $\lambda_1, ..., \lambda_n$ with λ_1 real, nonnegative and of maximum modulus ($\lambda_1 = |\lambda_1| \geq |\lambda_i|$ for $i = 1, 2, ..., n$)

Nonnegative realization has size at least:

- n, if $\sum\limits_{i=1}^{n} \lambda_i \geq 0$
- $n + \left\lceil -\frac{\sum_{i=1}^{n} \lambda_i}{\lambda_1} \right\rceil$, if $\sum\limits_{i=1}^{n} \lambda_i < 0$

Upper Bound on Minimal Realizations (1)

Focus: Nonnegative function $h[k] = \sum_{i=1}^{n} c_i \lambda_i^{k-1}$ for $k \geq 1$ ($h[k] = 0$, otherwise)

- Strictly proper rational transfer function

$$H(z) = \sum_{i=1}^{n} \frac{c_i z^{-1}}{1 - \lambda_i z^{-1}}$$

- Minimal diagonal realization

$$A_c = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}, \ b_c = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \ c_c^T = \begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix}$$

Fact: $H(z)$ has nonnegative realization (A, b, c^T) iff, for $\alpha > 0$, $\beta > 0$, $\beta H(\alpha z)$ has nonnegative realization (namely, $(\alpha^{-1}A, \alpha^{-1}b, \beta c)$).

WLOG: $h[k] = 1 + \sum_{i=2}^{n} c_i \lambda_i^{k-1} \geq 0$ for all $k \geq 1$, where $|\lambda_i| < 1$

Upper Bound on Minimal Realizations (2)

Theorem: Nonnegative impulse response $h[k]$ for $k \geq 1$

$$h[k] = 1 + \sum_{i=2}^{n} c_i \lambda_i^{k-1} \geq 0 \;, \quad |\lambda_i| < 1 \quad \text{for} \quad i = 2, 3, ..., n$$

There exists $N \geq 0$, such that $h[k+N]$ for $k \geq 1$ has nonnegative realization of dimension $2(n-1)$.

Sketch of Proof:

1. $h[k+N]$ can be written as

$$h[k+N] = \sum_{i=2}^{n} \frac{1}{n-1} \left[1 + c_i \lambda_i^N (n-1) \lambda_i^{k-1} \right]$$

2. Choose N so that $|c_i \lambda_i^N| \leq \frac{1}{n-1}$:

$$N = \left\lceil \max_{i \neq 1} \left\{ - \frac{\ln(|c_i|(n-1))}{\ln(|\lambda_i|)} \right\} \right\rceil$$

UPPER BOUND ON MINIMAL REALIZATIONS (3)

Theorem: Nonnegative impulse response $h[k]$ for $k \geq 1$ ($h[k] = 0$, otherwise)

$$h[k] = 1 + \sum_{i=2}^{n} c_i \lambda_i^{k-1} \geq 0 \ , \quad |\lambda_i| < 1 \text{ for } i = 2, 3, ..., n$$

has nonnegative realization of size $N + 2(n-1)$.

Hints:

1. Combine a realization of a *finite nonnegative* response $h[1], h[2], ..., h[N]$ with the $2(n-1)$ realization of $h[k+N]$
2. Combined realization

$$A = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{A_N} & 0 \\ \hline b_{2(n-1)} & 0 & A_{2(n-1)} \end{array}\right] \ , \ b = \left[\begin{array}{c} b_N \\ \hline 0 \end{array}\right] \ , \ c^T = \left[\begin{array}{c|c} c_N^T & c_{2(n-1)}^T \end{array}\right]$$

UPPER BOUND ON MINIMAL REALIZATIONS (4)

Example:

- Consider nonnegative response $h[k]$ for $k \geq 1$ ($h[k] = 0$, otherwise)

$$h[k] = 1 - \frac{1}{3}\left(-\frac{9}{10}\right)^{k-1} + \frac{1}{3}\left(-\frac{8}{9}\right)^{k-1} - \frac{1}{3}\left(-\frac{7}{8}\right)^{k-1}$$

- $N = 0$; upper bound is 6

Example:

- Consider nonnegative response $h[k]$ for $k \geq 1$ ($h[k] = 0$, otherwise)

$$h[k] = 1 + 5\left(-\frac{1}{8}\right)^{k-1} + 3\left(-\frac{1}{9}\right)^{k-1}$$

- $N = 2$; upper bound is 6

CONCLUSIONS

Results:

- Nonnegative realization of DT responses
- Criterion for MIMO systems
- Continuous-time extension
- Minimality results, bounds on minimality

Future Directions:

- MIMO realizability conditions in terms of maximum modulus pole(s)
- Classification of minimal nonnegative realizations
- Minimality, bounds in terms of spectral properties
- Continuous-time case (characterization, bounds)

GENERAL REALIZATION (SISO CASE)

Discrete-Time System

$$\begin{aligned} x[k+1] &= Ax[k] + bu[k] \\ y[k] &= c^T x[k] \end{aligned}$$

$$H(z) = c^T(zI - A)^{-1}b$$

$$h[k] = \begin{cases} c^T A^{k-1} b & k \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

Continuous-Time System

$$\begin{aligned} \dot{x}(t) &= Ax(t) + bu(t) \\ y(t) &= c^T x(t) \end{aligned}$$

$$H(s) = c^T(sI - A)^{-1}b$$

$$h(0) = \begin{cases} c^T e^{At} b & t > 0 \\ 0 & \text{otherwise} \end{cases}$$

OBSERVABILITY CANONICAL FORM (GENERAL REALIZATION)

Strictly proper rational transfer function

$$H(z) = \frac{p_1 z^{n-1} + p_2 z^{n-2} + ... + p_n}{z^n + q_1 z^{n-1} + ... + q_n} = \sum_{k=1}^{+\infty} h[k] z^{-k}$$

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -q_n & -q_{n-1} & -q_{n-2} & \dots & -q_1 \end{bmatrix}, \; b_c = \begin{bmatrix} h[1] \\ h[2] \\ \vdots \\ h[n] \end{bmatrix}, \; c_c^T = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}$$

Minimal realization iff $H(z)$ is reduced.

FURTHER GENERALIZATIONS (FARINA) (1)

Nonnegative impulse response $h[k] \geq 0$ for $k \geq 1$ ($h[k] = 0$, otherwise).
Strictly proper $H(z)$ with poles of maximum modulus at

$$\{\bar{\lambda}e^{j\theta_1}, \bar{\lambda}e^{j\theta_2}, ..., \bar{\lambda}e^{j\theta_k}\}$$

where $\bar{\lambda} > 0$ is the maximum modulus.

Definitions:

- The cyclicity of $H(z)$ is the smallest integer M, $\omega_i^M = 1$ for all $i = 1, 2, ..., k$
- *Cyclic* $H(z)$ if M exists
- *Primitive* $H(z)$ if $M = 1$

FURTHER GENERALIZATIONS (FARINA) (2)

Nonnegative impulse response $h[k] \geq 0$ for $k \geq 1$ ($h[k] = 0$, otherwise).
If M is the cyclicity of $h[k]$, we break it into the following M functions:

$$\begin{aligned} h_1[k] &= h[1 + (k-1)M] \\ h_2[k] &= h[2 + (k-1)M] \\ &\vdots \\ h_M[k] &= h[M + (k-1)M] \end{aligned}$$

We apply the same procedure to each $h_i[k]$ (or the corresponding $H_i(z)$) until we arrive at "primitive" functions (with cyclicity 1).

Theorem (Farina): $h[k]$ has nonnegative realization iff:

(i) $h[k]$ is nonnegative, and

(ii) All functions involved in the construction ($h[k]$, $h_i[k]$, $h_{i_j}[k]$, etc.) are cyclic or primitive

SUFFICIENCY OF VAN DEN HOF'S CONDITION

- $\mathcal{X} = cone(X)$ is backward-p-shift-invariant
- Satisfies $cone(H) \subset \mathcal{X}$

We would like to conclude that $H[k]$ has a *nonnegative realization*.
Sketch of Proof:

- Polyhedral $\Rightarrow X \subset R^{\infty \times n}$, such that $\mathcal{X} = cone(X)$
- $cone(H) \subset \mathcal{X} \Rightarrow H = XB$ for *nonnegative* B
- $\sigma^p(X) = XA$ for some *nonnegative* $n \times n$ matrix A
- C is a $p \times m$ matrix defined as $C(i,j) = X(i,j)$
- Can show that

$$S = \begin{bmatrix} C \\ CA \\ \vdots \end{bmatrix}$$

EXAMPLE: TWO-DIMENSIONAL CASE

Special case: Nonnegative impulse response $h[k]$ for $k \geq 1$ ($h[k] = 0$, otherwise)

$$h[k] = 1 + c_2 \lambda_2^{k-1} \ , \quad |\lambda_2| < 1$$

Corresponding diagonal realization:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \ , \ b = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \ , \ c^T = \begin{bmatrix} 1 & c_2 \end{bmatrix}$$

Restriction: $h[k] = c^T A^{k-1} b \geq 0$ for all $k \geq 1$.

Regardless of λ_2 and/or c_2, above realization can be *transformed* (using a suitable similarity transformation) into nonnegative realization.

Conclusion: $h[k]$ has a nonnegative realization (of size 2)

EXAMPLE: FINITE IMPULSE RESPONSE

A nonnegative finite response $h[k]$ with $h[k] = 0$ for $k < 1$ and for $k > N$

$h[k]$ has the following N-dimensional nonnegative realization

$$A_N = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \ b_N = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \ c_N^T = \begin{bmatrix} h[N] & h[N-1] & \dots & h[2] & h[1] \end{bmatrix}$$

UPPER BOUND ON MINIMAL REALIZATIONS

Example:

$$h[k] = 1 + 5\left(-\frac{1}{8}\right)^{k-1} + 3\left(-\frac{1}{9}\right)^{k-1}\ ,\ \text{for } k \geq 1$$

Actual nonnegative realization:

$$A = \left[\begin{array}{cc|cccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & \frac{1}{8} & 0 & 0 \\ 0 & 0 & 1 & \frac{7}{8} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & \frac{1}{9} \\ 0 & 0 & 0 & 0 & 1 & \frac{8}{9} \end{array}\right]\ ,\ b = \left[\begin{array}{c} 0 \\ 1 \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{array}\right]\ ,\ c = \left[\begin{array}{c} h[2] \\ h[1] \\ \hline \frac{1}{2} + c_2\lambda_2^2 \\ \frac{1}{2} + c_2\lambda_2^3 \\ \frac{1}{2} + c_3\lambda_3^2 \\ \frac{1}{2} + c_3\lambda_3^3 \end{array}\right] \equiv \left[\begin{array}{c} 0.0417 \\ 9.0000 \\ \hline 0.5781 \\ 0.4902 \\ 0.5370 \\ 0.4959 \end{array}\right]$$