FAULT-TOLERANT COMPUTATION IN SEMIGROUPS AND SEMIRINGS

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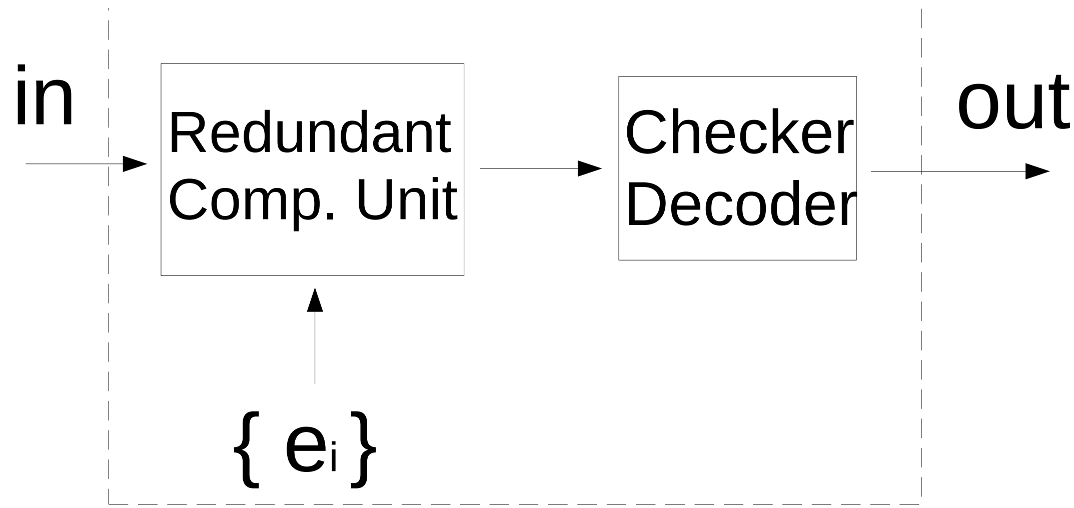
Outline

- Background
- Fault Tolerance for Semigroup-Based Computations
- Other Extensions and Future Possibilities
- Contribution

Fault-Tolerant Computational System



- Focus on "simple" binary operations: $out = in_1 \circ in_2$
- If same output and input spaces, then operation $\circ$ is a **semigroup** operation
- Special case when $\circ$ is a **group** operation (P. Beckmann, PhD Thesis, MIT, 1992)

Arithmetic Codes $\longleftrightarrow$ ABFT

(... Diamond, 1955 $\longleftrightarrow$ Abraham, 1984 ...)

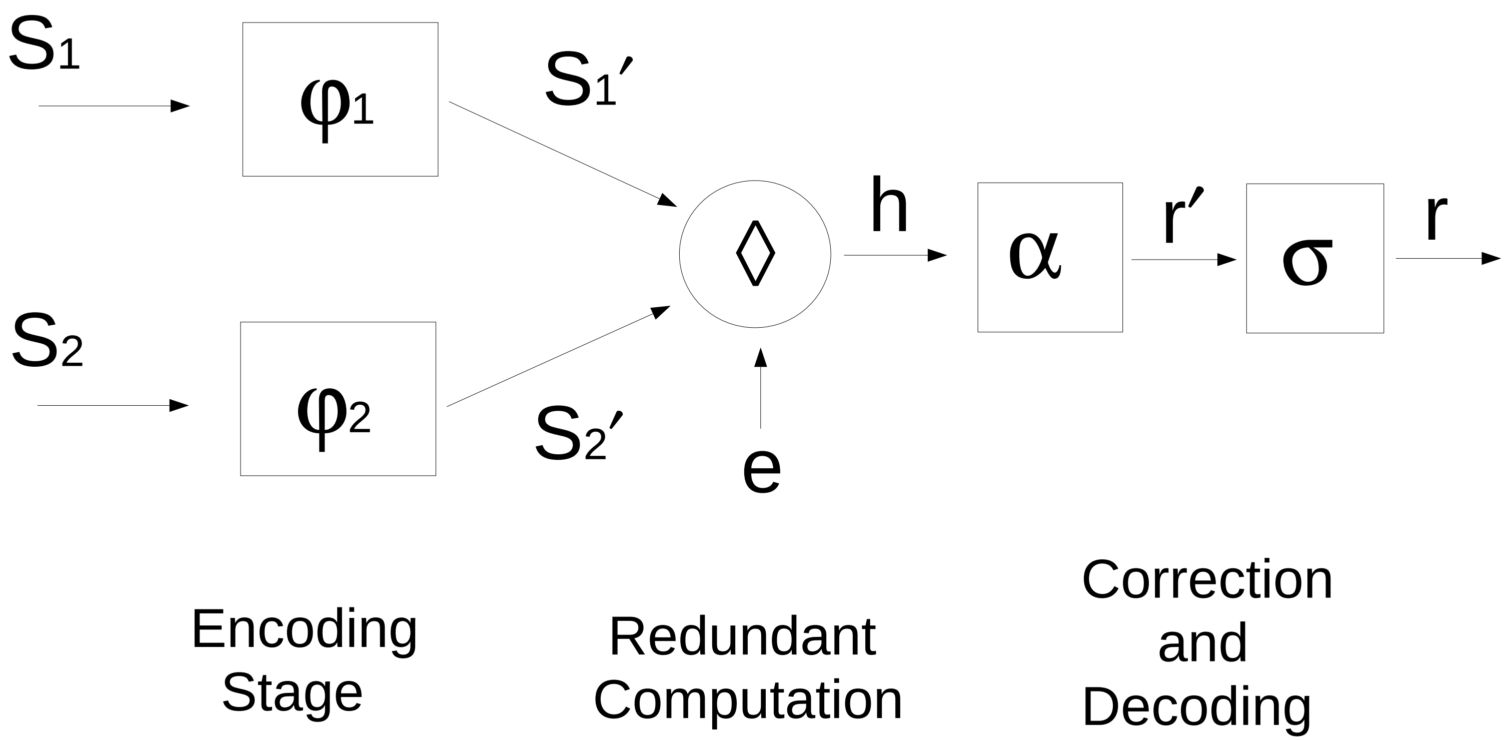
$n \times n$ Matrix Multiplication

(Abraham et al., 1984 onwards)

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{bmatrix}$$

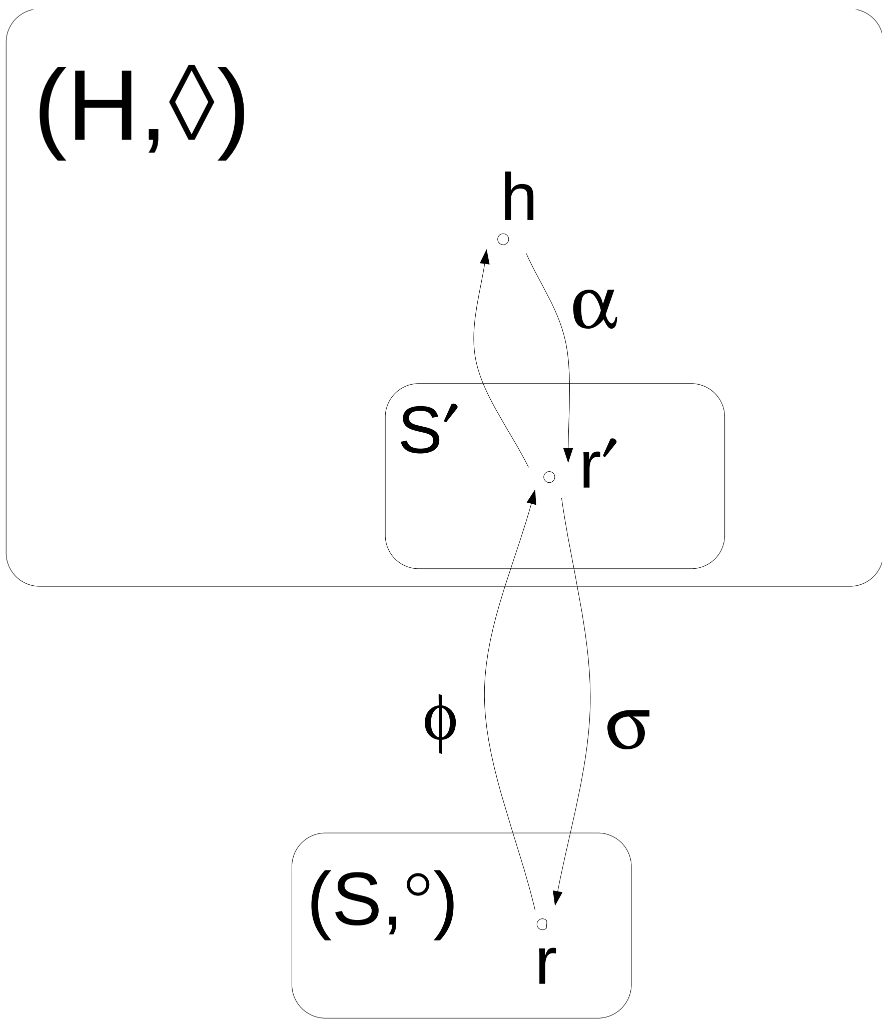
$$\begin{bmatrix} & A & & \\ c_1 & c_2 & \dots & c_n \end{bmatrix} \times \begin{bmatrix} & r_1 \\ & r_2 \\ B & \vdots \\ & r_n \end{bmatrix} = \begin{bmatrix} & & & & rs_1 \\ & & & & rs_2 \\ & & C & & \vdots \\ & & & & rs_n \\ cs_1 & cs_2 & \dots & cs_n & ? \end{bmatrix}$$

Fault-Tolerant Semigroup Computation



The computation to be protected is a **semigroup operation**:

Use of Abelian Semigroup Homomorphisms



Homomorphism: $\phi(s_1) \diamond \phi(s_2) = \phi(s_1 \circ s_2)$

Separate Codes

Redundant Semigroup given by "cartesian" product
$H = S \times T$.

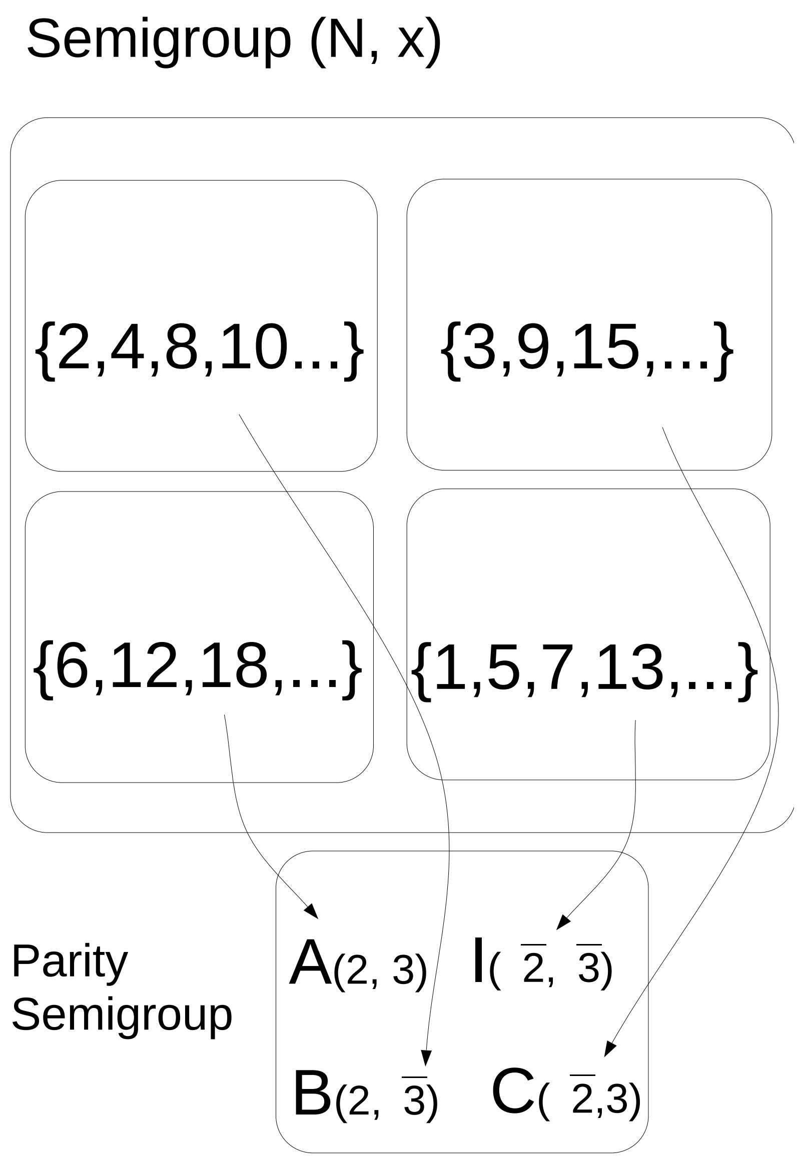
Semigroup T is used in a separate "parity" channel:

$$
\begin{aligned}
s_1 &\overset{\phi}{\longmapsto} (s_1, t_1) = (s_1, \theta(s_1)) \\
s_2 &\overset{\phi}{\longmapsto} (s_2, t_2) = (s_2, \theta(s_2)) \\
s_1 \circ s_2 &\overset{\phi}{\longmapsto} (s_1 \circ s_2, t_1 \odot t_2) = (s_1 \circ s_2, \theta(s_1 \circ s_2))
\end{aligned}
$$

where $\odot$ is the "parity" channel operation.

Parity encoding mapping θ is also a
semigroup homomorphism.

Separate Codes for $(N, \times)$



$\odot$	I	A	B	C
I	I	A	B	C
A	A	A	A	A
B	B	A	B	A
C	C	A	A	C

Extensions and Future Work

- More complicated algebraic structures can be studied. Already done for **semiring computations**, such as $(N_0, +, \times)$, and $(Z, max, +)$.

- Future Research

  1. Reflect hardware failure modes
  2. Study of error correction procedures (including distance metrics, syndromes, etc.)
  3. Protect strings of computations
  4. Links to error-correcting coding and automata theory

Contribution

- Algebraic framework for fault-tolerant semigroup computations

- Separate codes are completely characterized

- Possibilities for protecting non-linear signal processing applications