

Fault-Tolerant Linear Filters and Finite State Machines

Christoforos Hadjicostis

Decision and Control Group
Coordinated Science Laboratory

FAULT-TOLERANT SYSTEMS

Fault tolerance describes ability to

- Withstand internal failures
- Produce desirable overall “behavior”

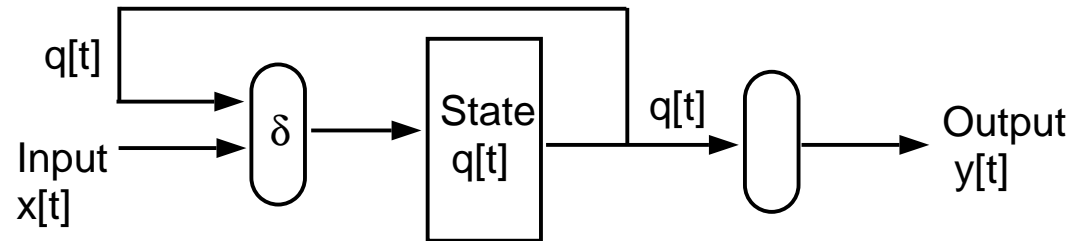
Necessary or desirable in

- Life-threatening circumstances (military, transportation, medical)
- Systems in inaccessible environments (space missions)
- Reliable systems from unreliable components
(faster, less expensive, less power)

RESEARCH ON FAULT TOLERANCE

- Communication systems (channel noise, error-correcting codes)
- Computational circuits (hardware failures, modular redundancy)
 - Each component fails with *constant* probability
 - Earlier work by von Neumann, Shannon, Winograd, Elias and others.
Later work by Gács, Pippenger, Feder, Reischuk.
- Special-purpose systems (Algorithm Based Fault Tolerance)
 - Protect against *fixed* number of failures
 - Recent work by Abraham et al., Redinbo, Musicus, Beckmann.
- Networked discrete-event systems (link or node failures, failure identification, reconfiguration/rerouting)

DYNAMIC SYSTEMS

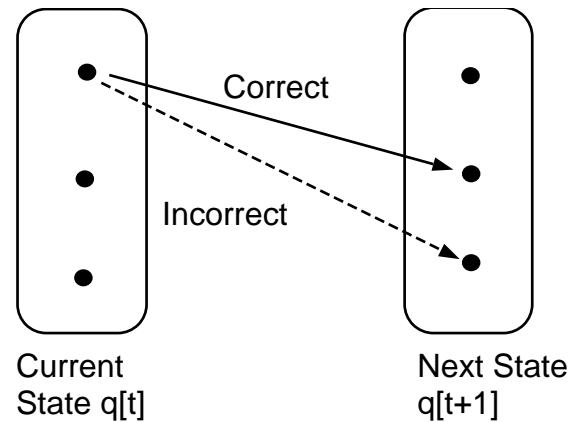


State Evolution: $q[t + 1] = \delta(q[t], x[t])$

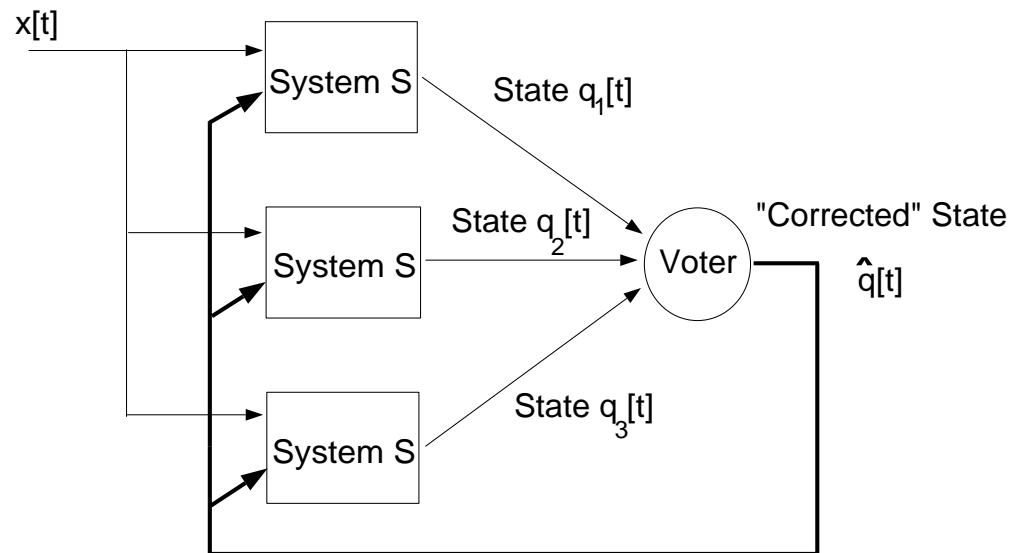
Output Equation: $y[t] = \lambda(q[t], x[t])$

Examples: digital filters, encoders/decoders, computer simulations

Failures in state transitions:



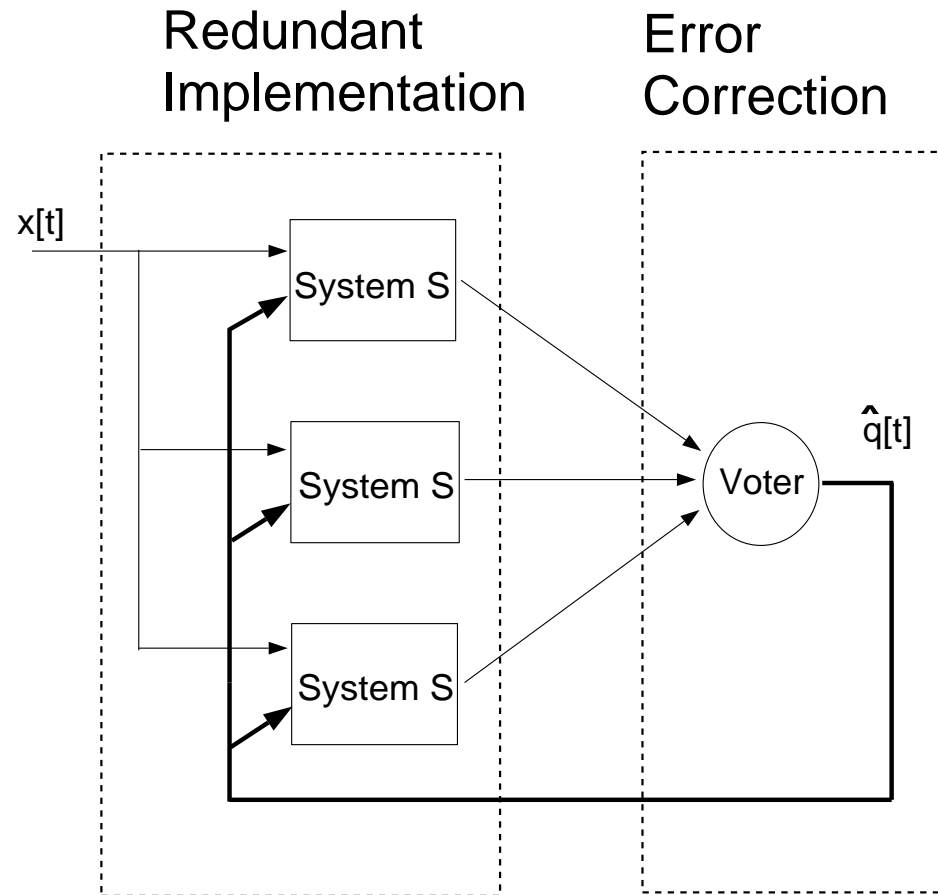
TRADITIONAL APPROACH: MODULAR REDUNDANCY (VON NEUMANN, 1956)



Problems:

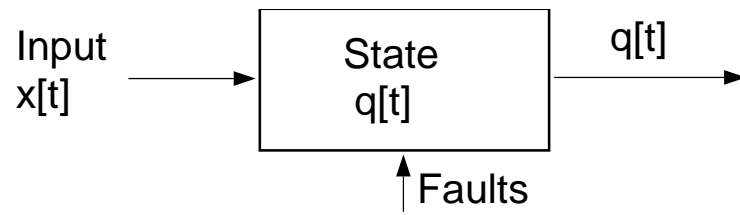
- Replication
- Voter failures

AVOIDING REPLICATION

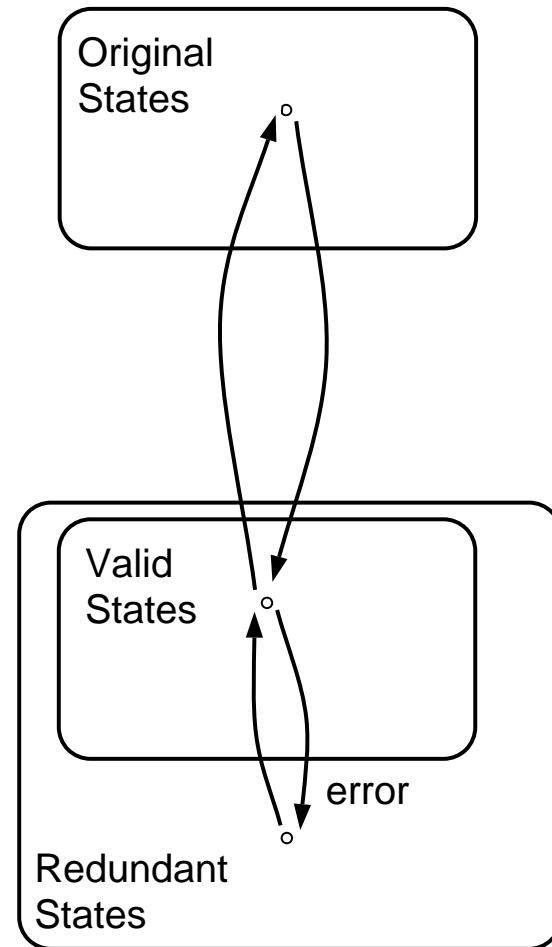
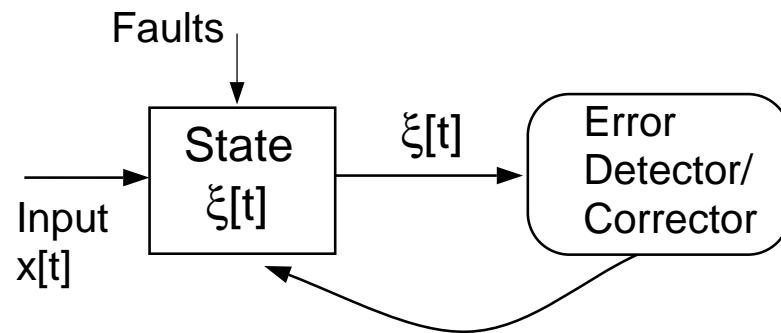


How can we avoid replication and minimize redundant hardware?

REDUNDANT IMPLEMENTATIONS



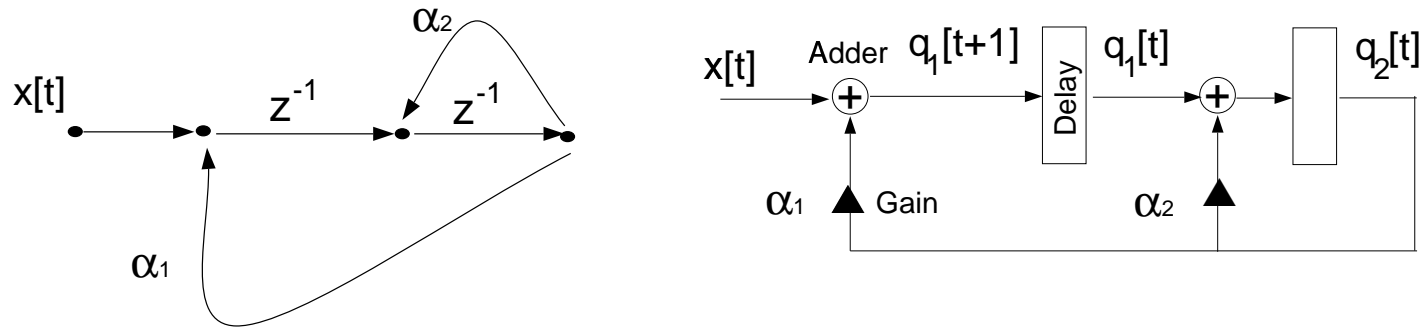
Replace with larger dynamic system:



DISCRETE-TIME LINEAR TIME-INVARIANT (LTI) DYNAMIC SYSTEMS

- Hardware implementation, error model
- Characterization of redundant implementations
 - “Equivalent” redundant implementations
- Encompasses Modular Redundancy and Checksum schemes
- New possibilities:
 - Exploit system dynamics to minimize hardware utilization
 - Development of linear coding schemes with “memory”
 - Reconfigurable decoding schemes

IMPLEMENTATIONS OF LTI DYNAMIC SYSTEMS

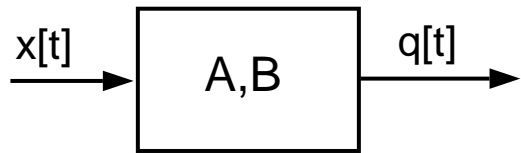


State Evolution: $\mathbf{q}[t + 1] = \mathbf{A}\mathbf{q}[t] + \mathbf{B}\mathbf{x}[t]$

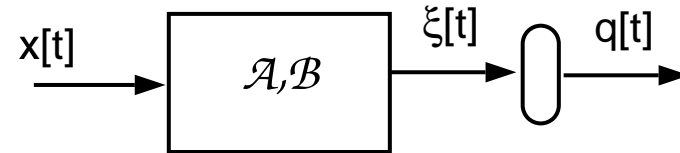
$$\mathbf{q}[t] \equiv \begin{bmatrix} q_1[t] \\ q_2[t] \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & \alpha_1 \\ 1 & \alpha_2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Given \mathbf{A} , \mathbf{B} , there exist many realizations (*signal flow graphs*)
- Focus on signal flow graphs with *delay-free paths of length one*:
 - Entries in \mathbf{A} , \mathbf{B} are values of gain elements
 - State variables in $\mathbf{q}[t + 1]$ are generated using *separate* gains and adders
 - A fault in an adder or in a gain element corrupts a *single* state variable

REDUNDANT LTI DYNAMIC SYSTEMS



$$\mathbf{q}[t + 1] = \mathbf{A}\mathbf{q}[t] + \mathbf{B}\mathbf{x}[t]$$



$$\xi[t + 1] = \mathcal{A}\xi[t] + \mathcal{B}\mathbf{x}[t]$$

- Concurrent Simulation: $\mathbf{q}[t] = \mathbf{L}\xi[t]$ (1)
- State Constraints: $\xi[t] = \mathbf{G}\mathbf{q}[t]$ (2)
- Fault Detection: When ξ is *not* in column space of \mathbf{G} , or $\mathbf{P}\xi[\cdot] \neq \mathbf{0}$
(where $\mathbf{P}\mathbf{G} = \mathbf{0}$)
- ◇ **Requirement:** \mathcal{A} and \mathcal{B} to be chosen appropriately to satisfy (1) and (2)

CHARACTERIZATION OF REDUNDANT LTI DYNAMIC SYSTEMS

$$\left. \begin{array}{l} \text{Original System} \\ \mathbf{q}[t + 1] = \mathbf{A}\mathbf{q}[t] + \mathbf{B}\mathbf{x}[t] \end{array} \right\} \begin{array}{l} \xi[\cdot] \xrightarrow{=\mathbf{G}\mathbf{q}[\cdot]} \\ \mathbf{q}[\cdot] \xleftarrow{=\mathbf{L}\xi[\cdot]} \end{array} \left\{ \begin{array}{l} \text{Redundant Implementation} \\ \xi[t + 1] = \mathcal{A}\xi[t] + \mathcal{B}\mathbf{x}[t] \end{array} \right.$$

Standard Redundant Implementations:

$(\mathcal{A}, \mathcal{B})$ is a redundant implementation for (\mathbf{A}, \mathbf{B}) iff $(\mathcal{A}, \mathcal{B})$ is similar to the following standard form:

$$\xi_\sigma[t + 1] = \underbrace{\begin{bmatrix} \mathbf{A} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{A}_{22} \end{bmatrix}}_{\mathcal{A}_\sigma} \xi_\sigma[t] + \underbrace{\begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix}}_{\mathcal{B}_\sigma} \mathbf{x}[t]$$

for some matrices $\mathbf{A}_{12}, \mathbf{A}_{22}$

Specifically: Invertible \mathcal{T} such that

$$\mathcal{A}_\sigma = \mathcal{T}^{-1} \mathcal{A} \mathcal{T}$$

Related work: Šiljak's "inclusion principle".

SIMILARITY TRANSFORMATIONS

Original System:

$$\begin{aligned}\mathbf{q}[t + 1] &= \mathbf{A}\mathbf{q}[t] + \mathbf{B}\mathbf{x}[t] \\ \mathbf{y}[t] &= \mathbf{C}\mathbf{x}[t] + \mathbf{D}\mathbf{x}[t]\end{aligned}$$

Similar System:

$$\begin{aligned}\mathbf{q}'[t + 1] &= (\mathbf{T}^{-1}\mathbf{A}\mathbf{T})\mathbf{q}'[t] + (\mathbf{T}^{-1}\mathbf{B})\mathbf{x}[t] \\ \mathbf{y}[t] &= (\mathbf{C}\mathbf{T})\mathbf{q}'[t] + \mathbf{D}\mathbf{x}[t]\end{aligned}$$

Both system have:

- Same input-output behavior
- Relationship between the two state vectors:

$$\mathbf{q}'[t] = \mathbf{T}^{-1}\mathbf{q}[t]$$

EXAMPLE: TRIPLE MODULAR REDUNDANCY

$$\xi[t+1] \equiv \begin{bmatrix} \mathbf{q}^1[t+1] \\ \mathbf{q}^2[t+1] \\ \mathbf{q}^3[t+1] \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A} \end{bmatrix} \xi[t] + \begin{bmatrix} \mathbf{B} \\ \mathbf{B} \\ \mathbf{B} \end{bmatrix} \mathbf{x}[t]$$

Possible decoding, encoding and parity check matrices are:

$$\mathbf{L} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \mathbf{I}_n \\ \mathbf{I}_n \\ \mathbf{I}_n \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} -\mathbf{I}_n & \mathbf{I}_n & \mathbf{0} \\ -\mathbf{I}_n & \mathbf{0} & \mathbf{I}_n \end{bmatrix}$$

Similar *standard* redundant implementation:

$$\xi_\sigma[t+1] = \left[\begin{array}{c|cc} \mathbf{A} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A} \end{array} \right] \xi_\sigma[t] + \left[\begin{array}{c} \mathbf{B} \\ \mathbf{0} \\ \mathbf{0} \end{array} \right] \mathbf{x}[t]$$

Corresponding *standard* decoding, encoding and parity check matrices:

$$\mathbf{L}_\sigma = \begin{bmatrix} \mathbf{I}_n & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{G}_\sigma = \begin{bmatrix} \mathbf{I}_n \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{P}_\sigma = \begin{bmatrix} \mathbf{0} & \mathbf{I}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_n \end{bmatrix}$$

EXAMPLE: FAULT-TOLERANT STATE FILTERS (CHATTERJEE AND D'ABREU, 1993)

Encode using a matrix \mathbf{C} (of dimension $d \times n$):

$$\xi[t+1] = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{CA} & \mathbf{0} \end{bmatrix} \xi[t] + \begin{bmatrix} \mathbf{B} \\ \mathbf{CB} \end{bmatrix} \mathbf{x}[t]$$

State constraints:

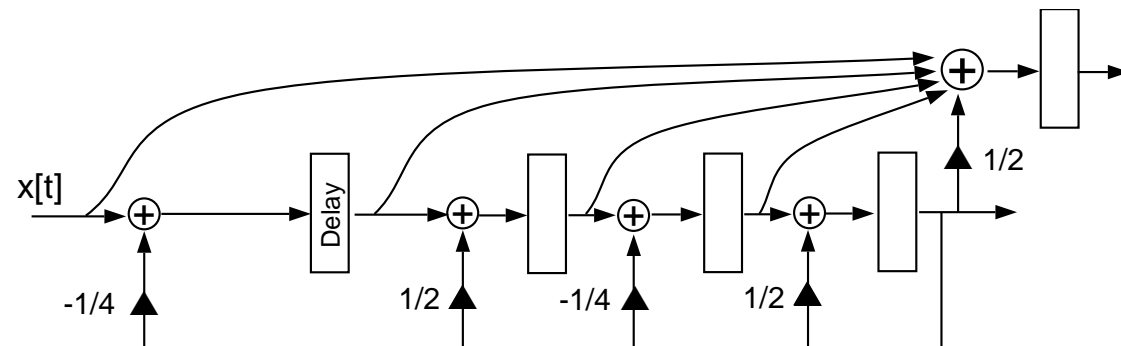
$$\xi[t] = \mathbf{G}\mathbf{q}[t] = \begin{bmatrix} \mathbf{I}_n \\ \mathbf{C} \end{bmatrix} \mathbf{q}[t] = \begin{bmatrix} \mathbf{q}[t] \\ \mathbf{C}\mathbf{q}[t] \end{bmatrix}$$

Similar *standard* redundant implementation (use $\mathcal{T} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{C} & \mathbf{I}_d \end{bmatrix}$, $\xi_\sigma = \mathcal{T}\xi[t]$):

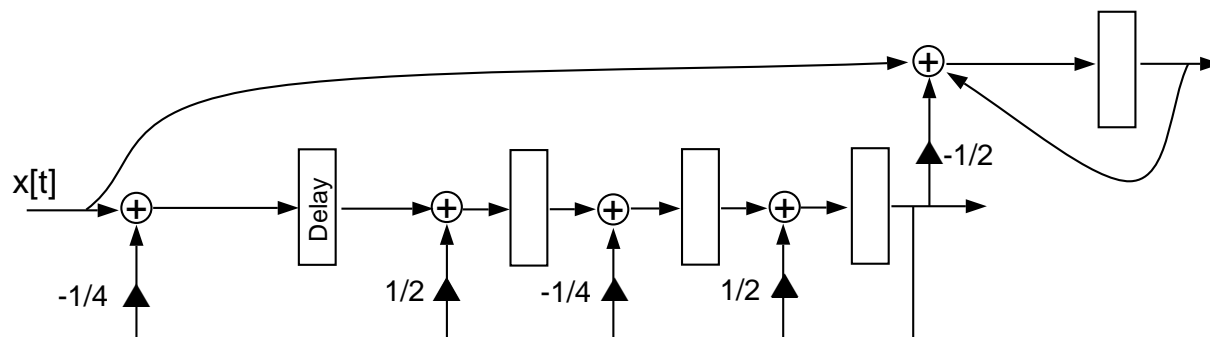
$$\xi_\sigma[t+1] = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array} \right] \xi_\sigma[t] + \left[\begin{array}{c} \mathbf{B} \\ \mathbf{0} \end{array} \right] \mathbf{x}[t]$$

DIFFERENT REDUNDANT IMPLEMENTATIONS FOR CHECKSUM SCHEME

Traditionally (Chatterjee and d'Abreu, Abraham et al., Reed):



Using previous theorem:



CONSTRUCTING REDUNDANT IMPLEMENTATIONS FOR LTI DYNAMIC SYSTEMS

1. Start from standard redundant implementation

2. Select a *suitable* parity check matrix \mathbf{P}

$$\text{Syndrome} = \mathbf{P}\xi_f[t] = \mathbf{P}(\xi[t] + \mathbf{e}[t]) = \mathbf{P}\mathbf{e}[t]$$

3. Choose \mathcal{T} such that $\mathbf{P}\mathcal{T} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_d \end{bmatrix}$

4. Develop a redundant implementation with

$$\mathcal{A} = \mathcal{T} \begin{bmatrix} \mathbf{A} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{A}_{22} \end{bmatrix} \mathcal{T}^{-1}, \quad \mathcal{B} = \mathcal{T} \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{L} = \mathcal{T}^{-1} \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \end{bmatrix}, \quad \mathbf{G} = \mathcal{T} \begin{bmatrix} \mathbf{I}_n \\ \mathbf{0} \end{bmatrix}$$

Advantage: Use the flexibility in the choice of \mathbf{A}_{12} and \mathbf{A}_{22} :

EXAMPLE: SINGLE ERROR CORRECTION (1)

Goal: Provide *single-error correction* capability to:

$$\mathbf{q}[t+1] = \underbrace{\begin{bmatrix} .2 & 0 & 0 & 0 \\ 0 & .5 & 0 & 0 \\ 0 & 0 & .1 & 0 \\ 0 & 0 & 0 & .6 \end{bmatrix}}_{\mathbf{A}} \mathbf{q}[t] + \underbrace{\begin{bmatrix} 2 \\ -1 \\ 1 \\ -2 \end{bmatrix}}_{\mathbf{b}} x[t]$$

Step 1: Consider a *standard* redundant implementation ($\mathbf{A}_{12} = \mathbf{0}$)

$$\xi_{\sigma}[t+1] = \left[\begin{array}{cccc|ccc} .2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & .5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .6 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & .2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & .5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & .1 \end{array} \right] \xi_{\sigma}[t] + \left[\begin{array}{c} 2 \\ -1 \\ 1 \\ -2 \\ 0 \\ 0 \\ 0 \end{array} \right] x[t]$$

with decoding, encoding and parity check matrices:

$$\mathbf{L}_{\sigma} = \begin{bmatrix} \mathbf{I}_4 & \mathbf{0} \end{bmatrix}, \quad \mathbf{G}_{\sigma} = \begin{bmatrix} \mathbf{I}_4 \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{P}_{\sigma} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_3 \end{bmatrix}$$

EXAMPLE: SINGLE ERROR CORRECTION (2)

Step 2: Desirable parity check matrix (Hamming code, locates single faults)

$$\mathbf{P} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Step 3: Find \mathcal{T} such that $\mathbf{P} = \mathbf{P}_\sigma \mathcal{T}$

Step 4: Obtain a similar (non-standard) redundant implementation:

$$\xi[t+1] = \begin{bmatrix} .2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & .5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .6 & 0 & 0 & 0 \\ 0 & -.3 & .1 & 0 & .2 & 0 & 0 \\ .3 & 0 & 0 & -.1 & 0 & .5 & 0 \\ -.1 & 0 & 0 & -.5 & 0 & 0 & .1 \end{bmatrix} \xi[t] + \begin{bmatrix} 2 \\ -1 \\ 1 \\ -2 \\ -2 \\ 1 \\ -1 \end{bmatrix} x[t]$$

- Extends to multiple faults (using other error-correcting or real number codes)

CHECKSUM SCHEMES WITH “MEMORY”

- *Temporal* failure corrupts the 4th state variable at time 0

$$\xi_f[0] = \xi[0] + \mathbf{e}[0] = \xi[0] + \begin{bmatrix} 0 \\ 0 \\ 0 \\ c \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Syndrome at time step 0

$$\mathbf{s}[0] \equiv \mathbf{P}\xi_f[0] = c \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

- Syndrome at time step m

$$\mathbf{s}[m] \equiv \mathbf{P}\xi[m] = c\mathbf{A}_{22}^m \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = c \begin{bmatrix} 0 \\ (.5)^m \\ (.1)^m \end{bmatrix}$$

EXAMPLE: RECONFIGURABLE DECODING (1)

The state evolution equation in the previous example was given by:

$$\xi[t + 1] = \begin{bmatrix} .2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & .5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .6 & 0 & 0 & 0 \\ 0 & -.3 & .1 & 0 & .2 & 0 & 0 \\ .3 & 0 & 0 & -.1 & 0 & .5 & 0 \\ .1 & 0 & .2 & -.3 & 0 & 0 & .3 \end{bmatrix} \xi[t] + \begin{bmatrix} 3 \\ -1 \\ 7 \\ 0 \\ -9 \\ -2 \\ -10 \end{bmatrix} x[t]$$

Observations:

- Permanent fault in second state variable ξ_2 at time step t propagates to fifth state variable (ξ_5) at time step $t + 1$
- Permanent fault in ξ_1 at time step t propagates to variables ξ_6 and ξ_7 at time step $t + 1$

EXAMPLE: RECONFIGURABLE DECODING (2)

Previously:

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & 0 & -1 \\ -1 & 0 & -1 & -1 \end{bmatrix}$$

Condition:
 $\mathbf{L}\mathbf{G} = \mathbf{I}_4$

When $\mathcal{A}(2, 2)$ is *permanently* corrupted, future $\xi_2[\cdot]$ and $\xi_5[\cdot]$ are invalid. Appropriate decoding and encoding mappings are as follows:

$$\mathbf{L}_r = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{G}_r = \begin{bmatrix} 1 & 0 & 0 & 0 \\ * & * & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ * & * & * & * \\ -1 & -1 & 0 & -1 \\ -1 & 0 & -1 & -1 \end{bmatrix}$$

Reconfigurability if:
 $\mathbf{L}_r\mathbf{G}_r = \mathbf{I}_4$

- Parity checks involving 2nd and 5th state variables are now *invalid*.

SUMMARY FOR LTI DYNAMIC SYSTEMS

- Error model, reflection of the design into hardware
- Encoding of LTI systems
 - Systematic introduction of redundancy
 - Generalizes/combines modular redundancy and checksum schemes
 - Allows adaptive schemes, checksum schemes with memory, schemes that require less hardware
- Standard form of redundant implementations;
non-zero redundant dynamics and coupling;

Related future work:

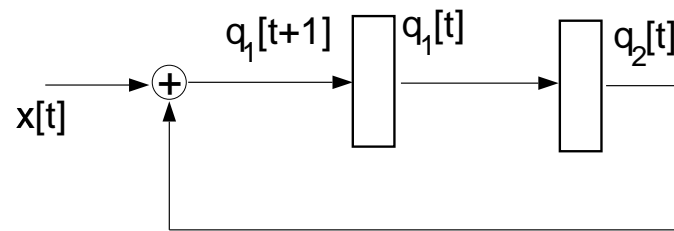
- General signal flow graphs (arbitrary implementations)
using factored state variables or other techniques
- Offer protection to analog systems

LINEAR FINITE STATE MACHINES

State evolution:

$$\mathbf{q}[t + 1] = \mathbf{A}\mathbf{q}[t] \oplus \mathbf{B}\mathbf{x}[t]$$

\mathbf{A} , \mathbf{b} , $\mathbf{q}[\cdot]$ and $x[\cdot]$ have entries in $GF(2)$

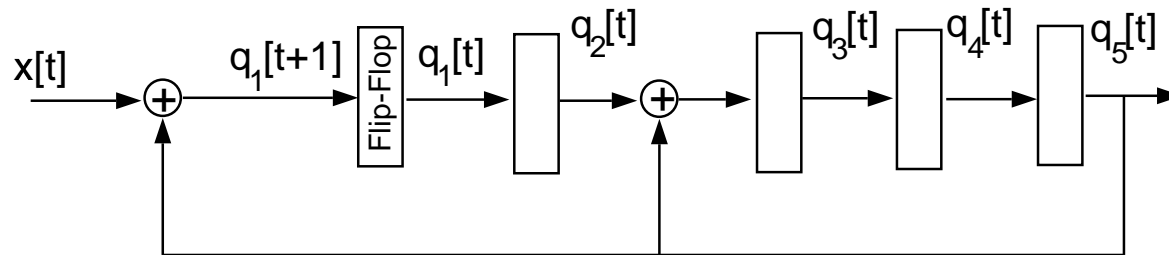


$$\mathbf{q}[t] \equiv \begin{bmatrix} q_1[t] \\ q_2[t] \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Examples:

- linear feedback shift registers, linear cellular automata
- counters (sequence generators), random number generators
- encoders/decoders

EXAMPLE: LINEAR FEEDBACK SHIFT REGISTER (1)



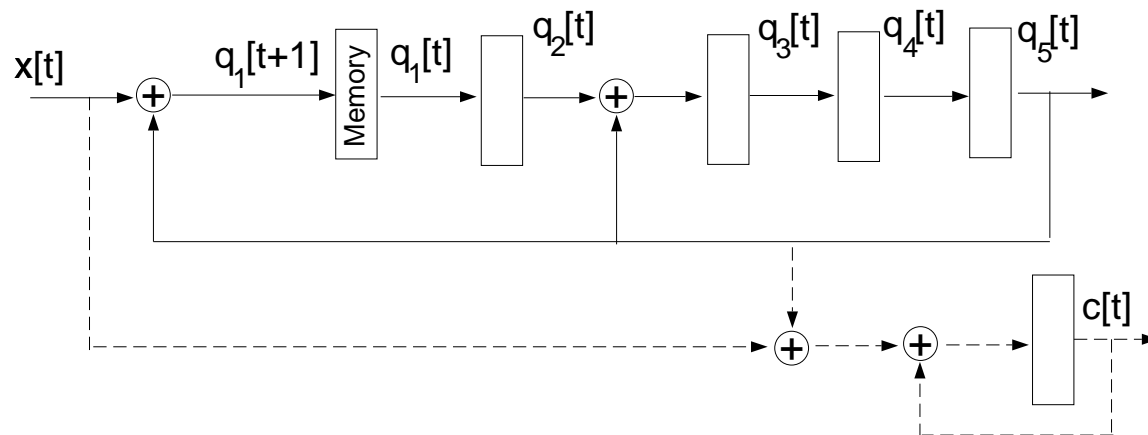
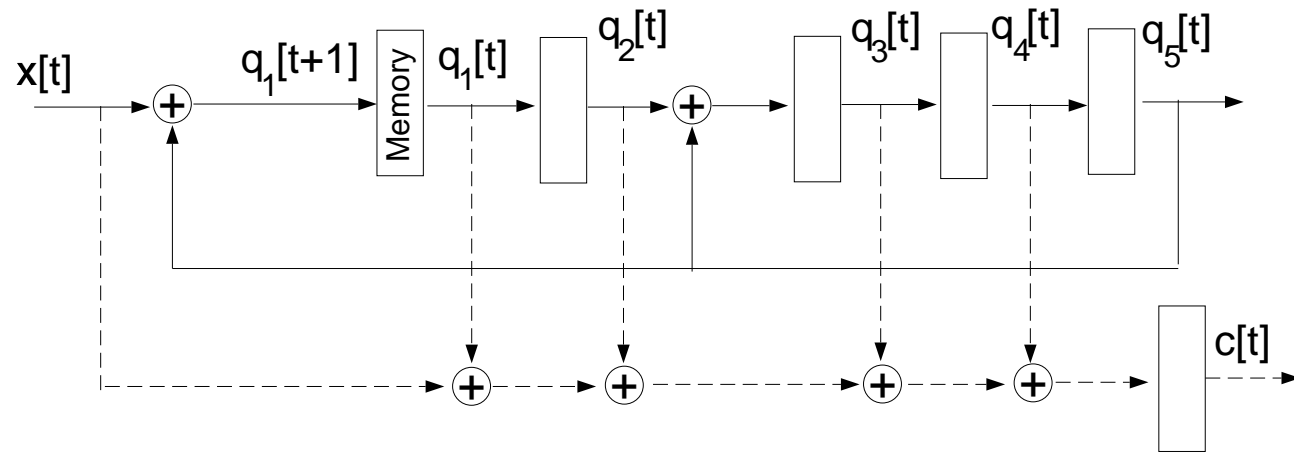
- State “evolution” $\mathbf{q}[t + 1] = \mathbf{A}\mathbf{q}[t] \oplus \mathbf{b}x[t]$ where

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Would like to enforce

$$\mathbf{G} = \begin{bmatrix} \mathbf{I}_5 \\ \mathbf{c}^T \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad \mathbf{L} = [\mathbf{I}_5 \quad \mathbf{0}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

EXAMPLE: LINEAR FEEDBACK SHIFT REGISTER (2)



REDUNDANT LINEAR FINITE STATE MACHINES

<u>Original LFSM</u> $\mathbf{q}[t + 1] = \mathbf{A}\mathbf{q}[t] \oplus \mathbf{B}\mathbf{x}[t]$ $(\mathbf{y}[t] = \dots)$ $\mathbf{q} \text{ is } n\text{-dimensional}$	$\xi[\cdot] \xrightarrow{=\mathbf{G}\mathbf{q}[\cdot]}$ $\mathbf{q}[\cdot] \xleftarrow{=\mathbf{L}\xi[\cdot]}$	<u>Redundant LFSM</u> $\xi[t + 1] = \mathcal{A}\xi[t] \oplus \mathcal{B}\mathbf{x}[t]$ $(\mathbf{y}[t] = \dots)$ $\xi \text{ is } \eta\text{-dimensional}$
--	---	---

- Encoding: $\xi[t] = \mathbf{G}\mathbf{q}[t]$
- Decoding: $\mathbf{q}[t] = \mathbf{L}\xi[t]$
- Fault detected when:
 - Redundant state vector is *not* in the column space of \mathbf{G}
 - $\mathbf{P}\xi[\cdot] \neq 0$ (where $\mathbf{P}\mathbf{G} = \mathbf{0}$)

CHARACTERIZATION OF REDUNDANT LFSM'S

THEOREM: Redundant LFSM's for an LFSM with state evolution:

$$\mathbf{q}[t + 1] = \mathbf{A}\mathbf{q}[t] \oplus \mathbf{B}\mathbf{x}[t]$$

are *similar* to

$$\xi_\sigma[t + 1] \equiv \begin{bmatrix} \xi_{\sigma_1}[t + 1] \\ \xi_{\sigma_2}[t + 1] \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{A}_{22} \end{array} \right] \xi_\sigma[t] \oplus \left[\begin{array}{c} \mathbf{B} \\ \mathbf{0} \end{array} \right] \mathbf{x}[t]$$

- Decoding: $\mathbf{q}[t] = \mathbf{L}_\sigma \xi_\sigma[t]$, where $\mathbf{L}_\sigma = \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \end{bmatrix}$
- Encoding: $\xi_\sigma[t] = \mathbf{G}_\sigma \mathbf{q}[t]$, where $\mathbf{G}_\sigma = \begin{bmatrix} \mathbf{I}_n \\ \mathbf{0} \end{bmatrix}$
- Parity Check Matrix: Check if $\mathbf{P}_\sigma \xi_\sigma[t] = \begin{bmatrix} 0 & \mathbf{I}_d \end{bmatrix} \xi_\sigma[t] = \mathbf{0}$

MINIMIZING COMPUTATIONAL HARDWARE

- For any given unconstrained LFSM, we characterize all embeddings that satisfy

$$\mathbf{G} = \begin{bmatrix} \mathbf{I}_n \\ \mathbf{C} \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} \mathbf{C} & \mathbf{I}_d \end{bmatrix},$$

(where \mathbf{C} is a $d \times n$ binary matrix)

- The redundant LFSM will have $\xi[t+1] = \mathcal{A}\xi[t] \oplus \mathcal{B}\mathbf{x}[t]$ where

$$\mathcal{A} = \mathcal{T}^{-1} \begin{bmatrix} \mathbf{A} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{A}_{12} \end{bmatrix} \mathcal{T} = \left[\begin{array}{c|c} \mathbf{A} \oplus \mathbf{A}_{12}\mathbf{C} & \mathbf{A}_{12} \\ \hline \mathbf{C}\mathbf{A} \oplus \mathbf{C}\mathbf{A}_{12}\mathbf{C} \oplus \mathbf{A}_{22}\mathbf{C} & \mathbf{C}\mathbf{A}_{12} \oplus \mathbf{A}_{22} \end{array} \right],$$

$$\mathcal{B} = \mathcal{T}^{-1} \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{B} \\ \mathbf{C}\mathbf{B} \end{bmatrix},$$

where $\mathcal{T} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{C} & \mathbf{I}_d \end{bmatrix}$.

- $2^{(n+d) \times d}$ different redundant versions (same $\mathbf{L}, \mathbf{G}, \mathbf{P}$)
- Efficient search algorithm to minimize computational hardware

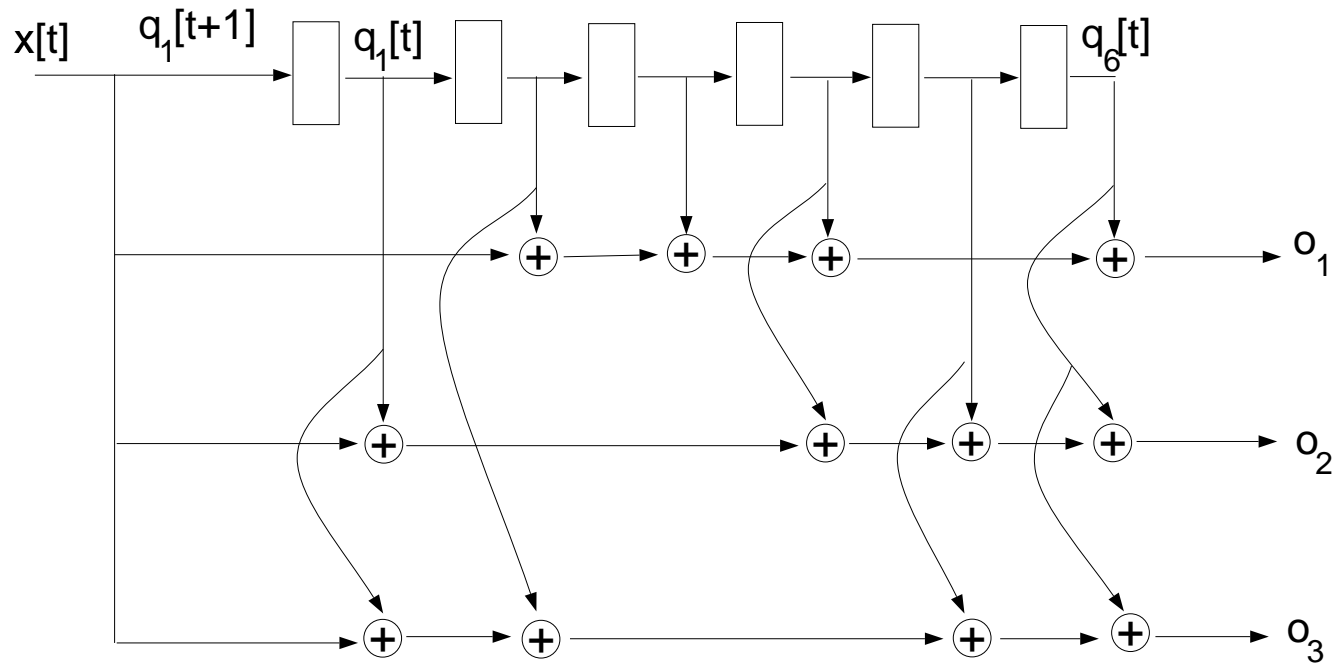
CONVOLUTIONAL ENCODER (1)

A convolutional encoder

$$g_1(D) = 1 \oplus D^2 \oplus D^3 \oplus D^4 \oplus D^6 \equiv (1011101)$$

$$g_2(D) = 1 \oplus D \oplus D^4 \oplus D^5 \oplus D^6 \equiv (1100111)$$

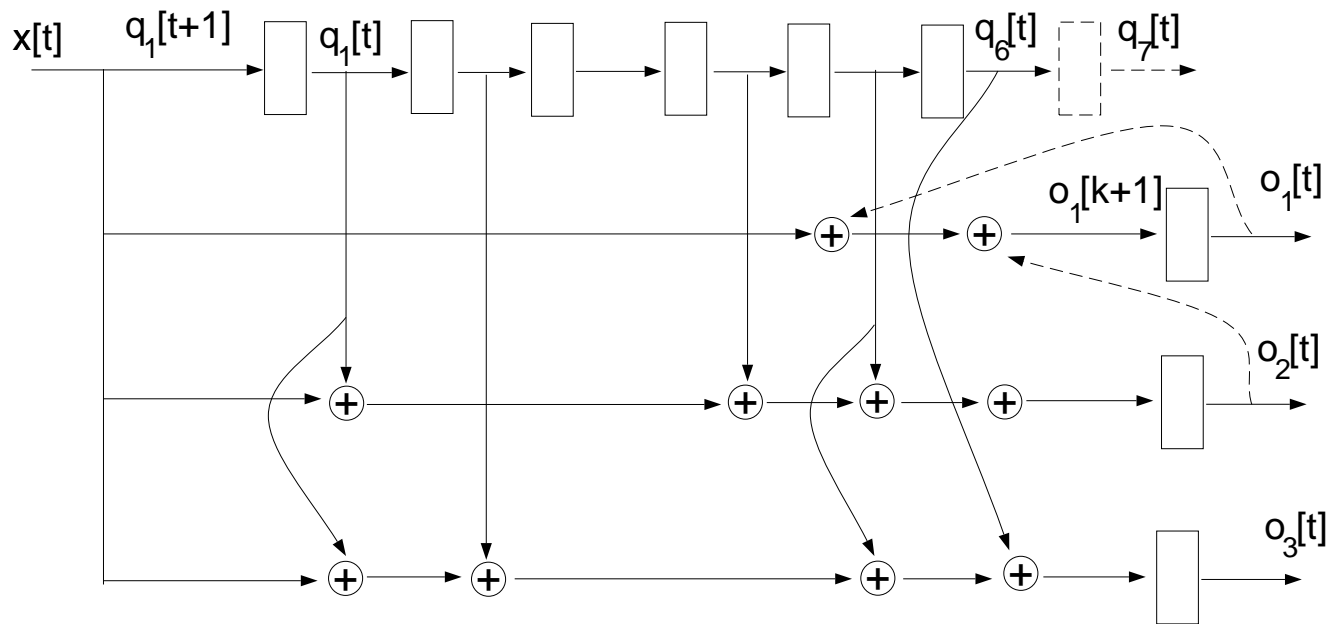
$$g_3(D) = 1 \oplus D \oplus D^2 \oplus D^5 \oplus D^6 \equiv (1110011)$$



CONVOLUTIONAL ENCODER (2)

- The encoder (that memorizes its outputs) has state of the form

$$\xi[k] \equiv \begin{bmatrix} \mathbf{q}[k] \\ \mathbf{o}[k] \end{bmatrix} = \begin{bmatrix} \mathbf{q}[k] \\ \mathbf{C}\mathbf{q}[k] \end{bmatrix}, \quad \text{where } \mathbf{C} = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$



CONCLUSIONS AND FUTURE WORK

- **Conclusions:**

- Applications to fault tolerance, reflection of hardware faults through appropriate error models
- Characterization of standard redundant dynamic systems (in LTI systems and linear FSM's)
- Efficient use of redundancy by exploiting algorithm or system structure

- **Future Work:**

- Protection schemes for discrete event systems
- Hierarchical and/or distributed error detection and correction
- Extensions to max-plus dynamic systems