Synthesis of switching rules for switched linear systems through randomized algorithms

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Introduction: Switched linear systems

- **Switched system with 2 subsystems:**

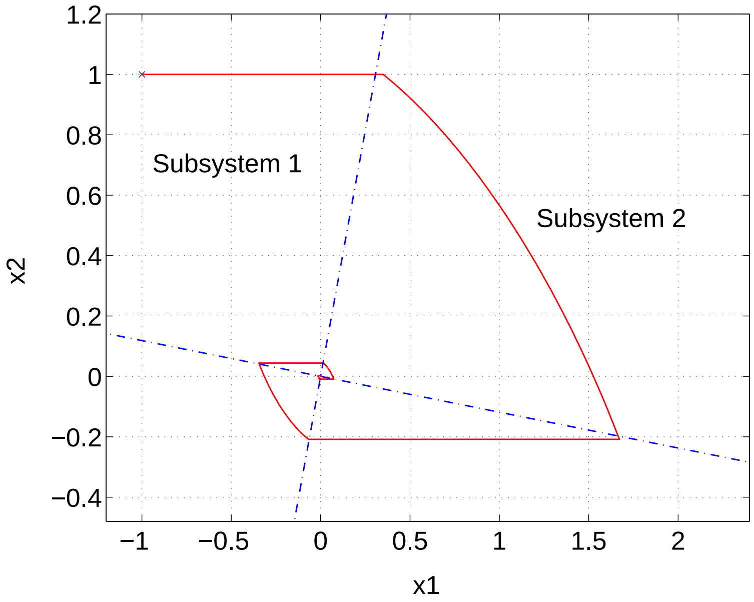
$$\dot{\boldsymbol{x}}(\boldsymbol{t}) = \boldsymbol{A}_{\boldsymbol{\sigma}(\boldsymbol{t})}\boldsymbol{x}(\boldsymbol{t})$$

  - $x(t) \in \mathbb{R}^n$
  - $\sigma(t) \in \{1, 2\}$: State of the switch
  - A_1 and A_2: Not Hurwitz
- **A hybrid system:** Continuous dynamics with a discrete signal
- How can we stabilize the system by switching?

Switching rule design

- Example:

$$A_1 = \begin{bmatrix} 0 & 10 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1.5 & 2 \\ -2 & -0.5 \end{bmatrix}$$



- Partition of $\mathbb{R}^n$: $\mathcal{X}_1$ and $\mathcal{X}_2$
- Switching rule: $\sigma(t) = j$, if $x(t) \in \mathcal{X}_j$ (State dependent)

Piecewise Lyapunov function

- Assume: $\alpha A_1 + (1-\alpha)A_2$ is not Hurwitz $\forall \alpha \in [0,1]$.

  (Otherwise, $\exists$ a switching rule with a quadratic Lyapunov function $V(x) = x'Px$.)

- **In this work:** Use of piecewise quadratic Lyapunov functions

$$V(x) = \max\{x'P_1x,\ x'P_2x\} \text{ with } P_j = P_j' \geq 0.$$

  - Many works on analysis via this approach
  - But few design techniques available

A sufficient condition

[Malmborg, Bernhardsson, Åström, 96]
[Wicks, DeCarlo, 97]

For $\gamma > 0$, suppose that $\forall x \in \partial\mathcal{B}_0 := \{x : \ \|x\| = 1\}$,

$$x'(P_1A_1 + A_1'P_1)x \le -\gamma, \quad \text{if } x'(P_1 - P_2)x \ge 0, \quad \text{and}$$
$$x'(P_2A_2 + A_2'P_2)x \le -\gamma, \quad \text{if } x'(P_1 - P_2)x \le 0.$$

Then, under switching with the partition

$$\mathcal{X}_1 = \{x : \ x'P_1x \ge x'P_2x\} \ \text{and} \ \mathcal{X}_2 = \{x : \ x'P_1x \le x'P_2x\},$$

we have a Lyapunov function $V(x) = \max\{x'P_1x, \ x'P_2x\}$.

This condition:

- Nonconvex: Checking for given P_1 and P_2 involves indef. quadratic programming (NP-complete).
- Equivalent to some bilinear matrix inequalities (BMI) via S-procedure

  (for 2-subsystem case only)
- Computationally difficult

In this work

- Develop a randomized algorithm that finds P_1 and P_2.
- Follow a probabilistic approach
  - proposed in [Polyak, Tempo, 2001] for a robust LQR problem;
  - effective for other control problems that are convex.
- Generalization to a special class of nonconvex problems
- Broader notion of convergence
- Can be extended to
  - General multi-modal case;
  - Switched nonlinear systems.
- Examples

Problem formulation

Let $\mathcal{C}^2 := \{(P_1, P_2) :\ P_j = P_j' \geq 0\}$.

Given $\gamma > 0$

Find $(P_1, P_2) \in \mathcal{C}^2$ such that $\forall x \in \partial\mathcal{B}_0$

$$\begin{aligned} x'(P_1A_1 + A_1'P_1)x &\leq -\gamma, \quad \text{if } x'(P_1 - P_2)x \geq 0, \quad \text{and} \\ x'(P_2A_2 + A_2'P_2)x &\leq -\gamma, \quad \text{if } x'(P_1 - P_2)x \leq 0. \end{aligned}$$

$\mathcal{P}_s$: Solution set

Assumptions

- Solution set $\mathcal{P}_s$ is nonempty.

- There exists $\epsilon > 0$ such that

$$\begin{aligned}\widetilde{\mathcal{P}}_{s,\epsilon} := \Big\{(P_1, P_2) \in \mathcal{C}^2 :\ & \text{for every } x \in \partial\mathcal{B}_0, \\ & x'(P_1A_1 + A_1'P_1)x \le -\gamma, \ \text{if } x'(P_1 - P_2)x \ge -\epsilon, \text{ and} \\ & x'(P_2A_2 + A_2'P_2)x \le -\gamma, \ \text{if } x'(P_1 - P_2)x \le \epsilon\Big\}\end{aligned}$$

is nonempty.

Solution set $\mathcal{P}_s$

Introduce the sets:

$$\mathcal{P}_1(x) := \Big\{(P_1, P_2) :\ x'(P_1 - P_2)x \geq 0\Big\},$$
$$\mathcal{P}_2(x) := \Big\{(P_1, P_2) :\ x'(P_1 - P_2)x \leq 0\Big\},$$
$$\mathcal{Q}_j(x) := \Big\{(P_1, P_2) :\ x'(P_jA_j + A_jP_j)x \leq -\gamma\Big\},\quad j = 1, 2.$$

Lemma: (i) $\mathcal{P}_j(x)$, $j = 1, 2$, partition $\mathcal{C}^2$ for each x.

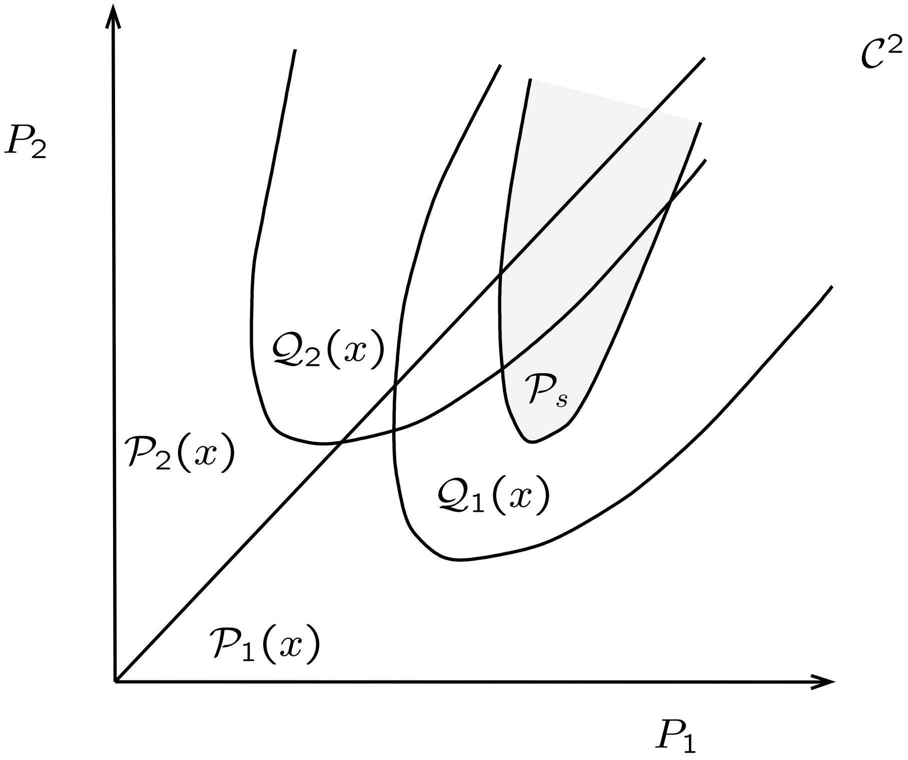
(ii) $$\mathcal{P}_s = \Big\{(P_1, P_2) :\ \ (P_1, P_2) \in \mathcal{P}_j(x) \ \Rightarrow\ (P_1, P_2) \in \mathcal{Q}_j(x), \quad \forall x \in \partial\mathcal{B}_0,\ \forall j \in \{1, 2\}\Big\}$$

$$= \bigcap_{x \in \partial\mathcal{B}_0} \bigcup_{j=1,2} \Big[\mathcal{P}_j(x) \cap \mathcal{Q}_j(x)\Big]$$

Sketch of the sets

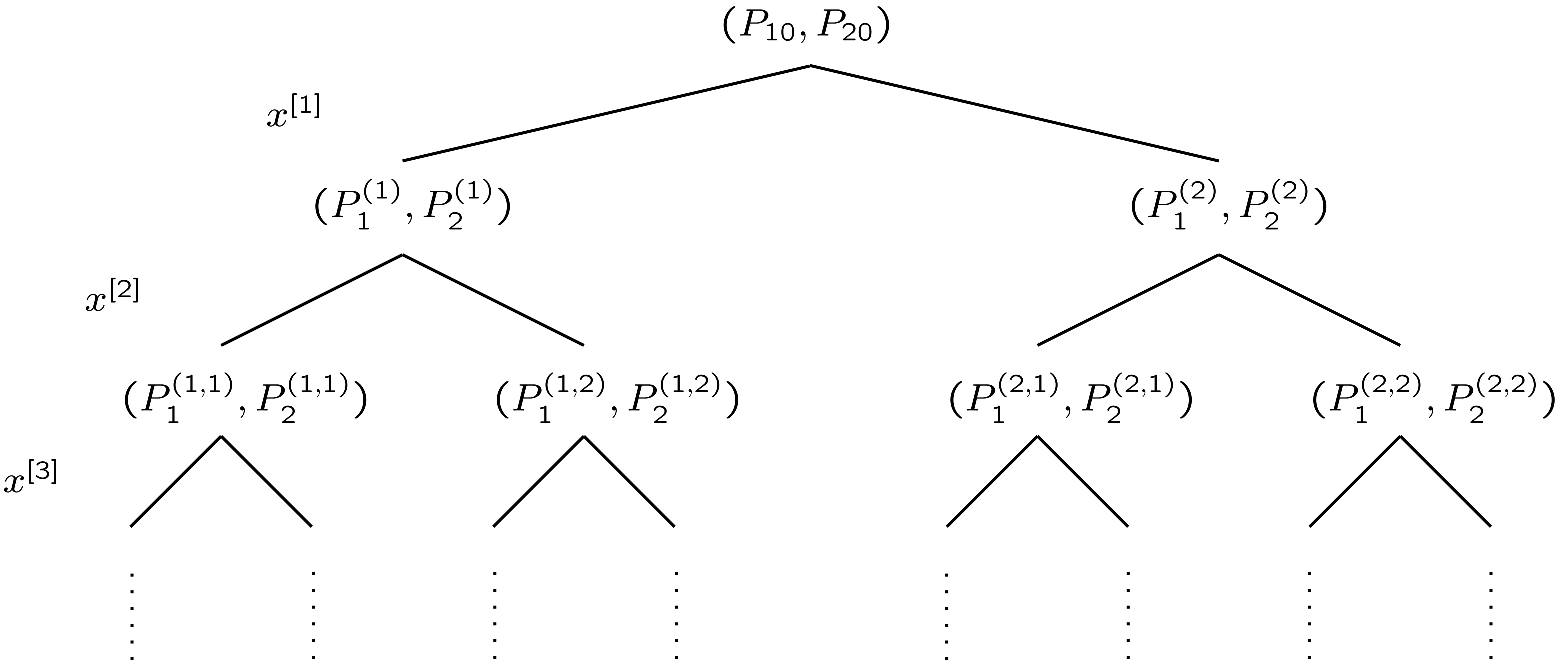
For each $x \in \partial\mathcal{B}_0$,

- $\mathcal{P}_s \subset \bigcup_{j=1,2}\left[\mathcal{P}_j(x) \cap \mathcal{Q}_j(x)\right]$
- $\mathcal{P}_1(x)$, $\mathcal{P}_2(x)$: Cones and contain 0
- $\mathcal{Q}_1(x)$, $\mathcal{Q}_2(x)$, and $\mathcal{P}_s$: Contain cones



Construction of gradient-based algorithms

- Given (P_1, P_2) and info on $\mathcal{P}_1(x) \cap \mathcal{Q}_1(x)$ and $\mathcal{P}_2(x) \cap \mathcal{Q}_2(x)$, to which direction shall (P_1, P_2) be heading?



(P_{10}, P_{20})

$x^{[1]}$

$(P_1^{(1)}, P_2^{(1)})$ $(P_1^{(2)}, P_2^{(2)})$

$x^{[2]}$

$(P_1^{(1,1)}, P_2^{(1,1)})$ $(P_1^{(1,2)}, P_2^{(1,2)})$ $(P_1^{(2,1)}, P_2^{(2,1)})$ $(P_1^{(2,2)}, P_2^{(2,2)})$

$x^{[3]}$

Rewrite the problem

- For $(P_1, P_2) \in \mathcal{C}^2$ and $x \in \partial\mathcal{B}_0$, let

$$v_1(P_1, P_2, x) := \{[-x'(P_1 - P_2)x]^+\}^2 + \{[x'(P_1A_1 + A_1'P_1)x + \gamma]^+\}^2,$$
$$v_2(P_1, P_2, x) := \{[-x'(P_2 - P_1)x]^+\}^2 + \{[x'(P_2A_2 + A_2'P_2)x + \gamma]^+\}^2,$$

where $[z]^+ = z$ if $z > 0$ and $[z]^+ = 0$ if $z \leq 0$.

  - Convex in (P_1, P_2) for fixed x
  - $v_j(P_1, P_2, x) \leq 0 \quad \Leftrightarrow \quad (P_1, P_2) \in \mathcal{P}_j(x) \cap \mathcal{Q}_j(x)$

- **Problem$'$:** Given $\gamma > 0$, find $(P_1, P_2) \in \mathcal{C}^2$ such that $\forall x \in \partial\mathcal{B}_0$,

  either $v_1(P_1, P_2, x) \leq 0$ or $v_2(P_1, P_2, x) \leq 0$.

- Pdf for random generation of states x: $f_x(x) > 0, \forall x \in \partial\mathcal{B}_0$

Randomized algorithm

Step 0: Set an initial pair $(P_{10}, P_{20}) \in \mathcal{C}^2$ and $r \in (0, \epsilon/4]$.

Step 1: (i) Generate a random state $x^{[1]} \in \partial\mathcal{B}_0$.

(ii) Obtain two pairs $(P_1^{(j)}, P_2^{(j)})$, $j = 1, 2$:

$$\left(P_1^{(j)}, P_2^{(j)}\right) = \begin{cases} \left([P_{10} - \mu^{(j)}\nabla_1 v^{(j)}]^+, [P_{20} - \mu^{(j)}\nabla_2 v^{(j)}]^+\right), & \\ & \text{if } v_j(P_{10}, P_{20}, x^{[1]}) > 0 \\ & \text{(i.e., } (P_{10}, P_{20}) \notin \mathcal{P}_j(x^{[1]}) \cap \mathcal{Q}_j(x^{[1]})\text{)}, \\ (P_{10}, P_{20}), & \text{otherwise,} \end{cases}$$

where $\nabla_i v^{(j)} := \nabla_{P_i} v_j(P_{10}, P_{20}, x^{[1]})$, $i = 1, 2$, the step sizes

$$\mu^{(j)} := \frac{v_j(P_{10}, P_{20}, x^{[1]}) + r\|(\nabla_1 v^{(j)}, \nabla_2 v^{(j)})\|}{\|(\nabla_1 v^{(j)}, \nabla_2 v^{(j)})\|^2}, \quad j = 1, 2,$$

and $[X]^+ = \arg\min_{Y \geq 0} \|X - Y\|$.

Randomized algorithm (cont'd)

Step 2: (i) Generate a random state $x^{[2]} \in \partial\mathcal{B}_0$.

(ii) Obtain four pairs $(P_1^{(j_1,j_2)}, P_2^{(j_1,j_2)})$, $j_1, j_2 = 1, 2$.

⋮

Step k: (i) Generate $x^{[k]} \in \partial\mathcal{B}_0$.

(ii) Obtain 2^k pairs $(P_1^{(j_1,\ldots,j_k)}, P_2^{(j_1,\ldots,j_k)})$, $j_1, \ldots, j_k = 1, 2$.

Theorem: There is $k \in \mathbb{N}$ such that one of the pairs in

$$\{(P_1^{(j_1,\ldots,j_k)}, P_2^{(j_1,\ldots,j_k)}) : \ j_1, \ldots, j_k = 1, 2\}$$

is a solution with probability one.

Outline of proof

1. There is a ball $\mathcal{B}_{(P_1^*, P_2^*)}(r) \subset \mathcal{C}^2$ contained in $\mathcal{P}_s$.

2. Suppose $(P_{10}, P_{20}) \notin \mathcal{P}_s$. At Step 1, there exists $j_1^* \in \{1, 2\}$ such that

(i) $\mathcal{P}_s \cap [\mathcal{P}_{j_1^*}(x^{[1]}) \cap \mathcal{Q}_{j_1^*}(x^{[1]})] \neq \emptyset$;

(ii) if there is an update in obtaining $(P_1^{(j_1^*)}, P_2^{(j_1^*)})$,

$$\|(P_1^{(j_1^*)}, P_2^{(j_1^*)}) - (P_1^*, P_2^*)\|^2 \leq \|(P_{10}, P_{20}) - (P_1^*, P_2^*)\|^2 - r^2;$$

(iii) otherwise, $(P_1^{(j_1^*)}, P_2^{(j_1^*)}) \in \mathcal{P}_{j_1^*}(x^{[1]}) \cap \mathcal{Q}_{j_1^*}(x^{[1]})$.

3. The pairs corresponding to $(j_1^*, \ldots, j_k^*)$, $k \in \mathbb{N}$, go closer to the ball monotonically.

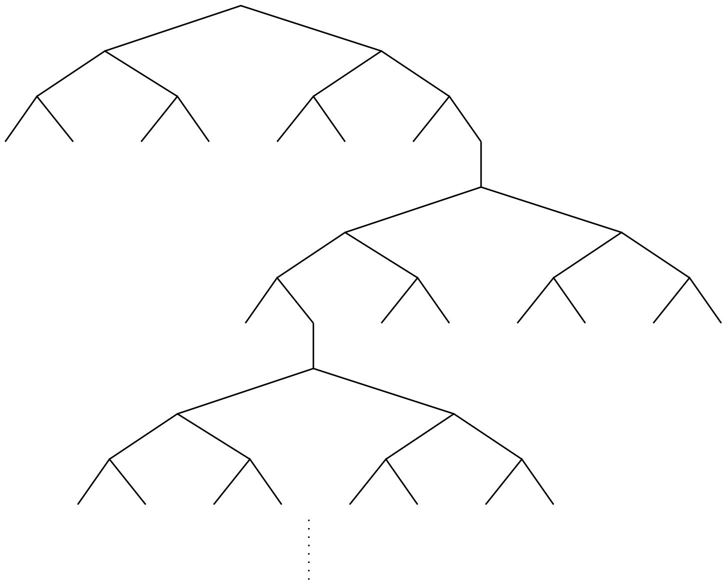
4. While such a pair is not a solution, the probability to generate x that forces an update is positive.

Remarks

- Convergence relies on convexity of $v_j(\cdot, \cdot, x)$
- Upper bound on # of updates
- The notion of convergence: More general than that for convex problems
- The number of candidate pairs: 2^k at step k

Modified algorithm using branch and bound

Practical version of the randomized algorithm



Branch: Generate 2^M pairs using the original algorithm.

Bound: Choose the pair that is “closest” to the solution set.

Modified algorithm using branch and bound (cont'd)

- $\{x^{[l]}\}_{l=1}^{N}$: Fixed set of random states with a large N

- **Empirical probability:**

$$\begin{aligned}
&\widehat{p}_N\,(P_1, P_2) \\
&\quad := \frac{1}{N}\Big|\Big\{x^{[l]} :\ (P_1, P_2) \in \bigcup_{j=1,2} \Big[\mathcal{P}_j(x^{[l]}) \cap \mathcal{Q}_j(x^{[l]})\Big]\,, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad l = 1, 2, \ldots, N\Big\}\Big|,
\end{aligned}$$

  where $|\cdot|$ is cardinality of a set.

- Bound: Choose the pair with the largest empirical probability.

- N may be chosen based on Chernoff bounds.

Extension to the multi-modal case

$$\dot{x}(t) = A_{\sigma(t)} x(t), \ \ \text{where } x(t) \in \mathbb{R}^n \text{ and } \sigma(t) \in \{1, \dots, L\}$$

- A_j, $j = 1, \dots, L$: Non-Hurwitz, and no Hurwitz convex combination
- **Goal:** Construct a stabilizing switching rule

$$\sigma(t) = j, \ \ \text{if } x(t)' P_j x(t) = \max_i x(t)' P_i x(t).$$

- **Problem:** Find L matrices $P_1, \dots, P_L \geq 0$ such that

  $\forall x \in \partial\mathcal{B}_0$, $\forall j \in \{1, \dots, L\}$,

$$x' P_j x = \max_i x' P_i x \ \ \Rightarrow \ \ x'(P_j A_j + A_j' P_j) x \leq -\gamma.$$

- Piecewise quadratic Lyapunov function $V(x) := \max_i \, x' P_i x$

Extension to the multi-modal case (cont'd)

- Convert the problem to

Find $(P_1, \ldots, P_L)$ such that

$$\forall x \in \mathcal{X},\ \exists j \in \{1, \ldots, L\} \text{ s.t. } v_j(P_1, \ldots, P_L,\ x) \leq 0, \qquad (*)$$

where $\mathcal{X} := \partial \mathcal{B}_0$ and

$$\begin{aligned} v_j(P_1, \ldots, P_L, x) := \Sigma_{i \neq j} \{[-x'(P_j - P_i)x]^+\}^2 \\ + \{[x'(P_j A_j + A_j' P_j)x + \gamma]^+\}^2. \end{aligned}$$

- $v_j(P_1, \ldots, P_L,\ x)$ is convex in $(P_1, \ldots, P_L)$ for fixed x.

- Can develop a randomized algorithm with probability-one convergence.

- No conservatism introduced (c.f., BMI approach).

Extension to the multi-modal case (cont'd)

Assumptions: $\mathcal{P}_s$ is nonempty, and there exists $\epsilon > 0$ such that

$$\widetilde{\mathcal{P}}_{s,\epsilon} := \Big\{(P_1,\dots,P_L) \in \mathcal{C}^L : \forall x \in \partial\mathcal{B}_0,\ \forall j \in \{1,\dots,L\}, \\ \text{if } x'P_jx \geq V(x) - \epsilon, \text{ then } x'(P_jA_j + A_jP_j)x \leq -\gamma\Big\}$$

is nonempty.

Algorithm: Step 0: Set $r \in (0, \epsilon/4]$ and $P_0 = (P_{10},\dots,P_{L0}) \in \mathcal{C}^L$.

Step 1: (i) Generate $x^{[1]} \in \partial\mathcal{B}_0$ based on pdf f_x.

(ii) Obtain L sequences $P^{(j)} \in \mathcal{C}^L$, $j = \{1,\dots,L\}$:

$$P^{(j)} = \begin{cases} \left[P_0 - \mu^{(j)} \nabla_P v_j(P_0, x^{[1]})\right]^+, & \text{if } v_j(P_0, x^{[1]}) > 0, \\ P_0, & \text{otherwise,} \end{cases}$$

where $\mu^{(j)} = (v_j(P_0, x^{[1]}) + r\|\nabla_P v_j(P_0, x^{[1]})\|)/\|\nabla_P v_j(P_0, x^{[1]})\|^2$.

Step k: (i) Generate $x^{[k]} \in \partial\mathcal{B}_0$.

(ii) Obtain L^k sequences $P^{(j_1,\dots,j_k)} \in \mathcal{C}^L$, $j_1,\dots,j_k = 1,\dots,L$.

Extension to switched nonlinear systems

- The system: $\dot{x}(t) = f_{\sigma(t)}(x(t))$ with $x(t) \in \mathbb{R}^n$ and $\sigma(t) \in \{1, 2\}$
- $f_j : \mathcal{X} \to \mathbb{R}^n$, $j = 1, 2$:
  - Locally Lipschitz with a bounded domain $\mathcal{X} \subset \mathbb{R}^n$ containing 0
  - Equilibrium 0 for each subsystem $\dot{x} = f_j(x)$
- Find (P_1, P_2) such that $\forall x \in \mathcal{X}$, $\exists j \in \{1, 2\}$ s.t. $v_j(P_1, P_2,\ x) \leq 0$, where

$$v_j(P_1, P_2, x) := \{[-x'(P_j - P_i)x]^+\}^2 + \{[2x'P_j f_j(x) + \gamma x'x]^+\}^2, \\ i, j = 1, 2, \quad i \neq j.$$

- Local asymptotic stability in a level set $\{x :\ V(x) \leq c\} \subset \mathcal{X}$ with some $c > 0$
- Assumptions and algorithms can be developed in a parallel manner.

Example 1: 2-dim. 2-subsystem case

$$A_1 = \begin{bmatrix} 0 & 10 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1.5 & 2 \\ -2 & -0.5 \end{bmatrix}$$

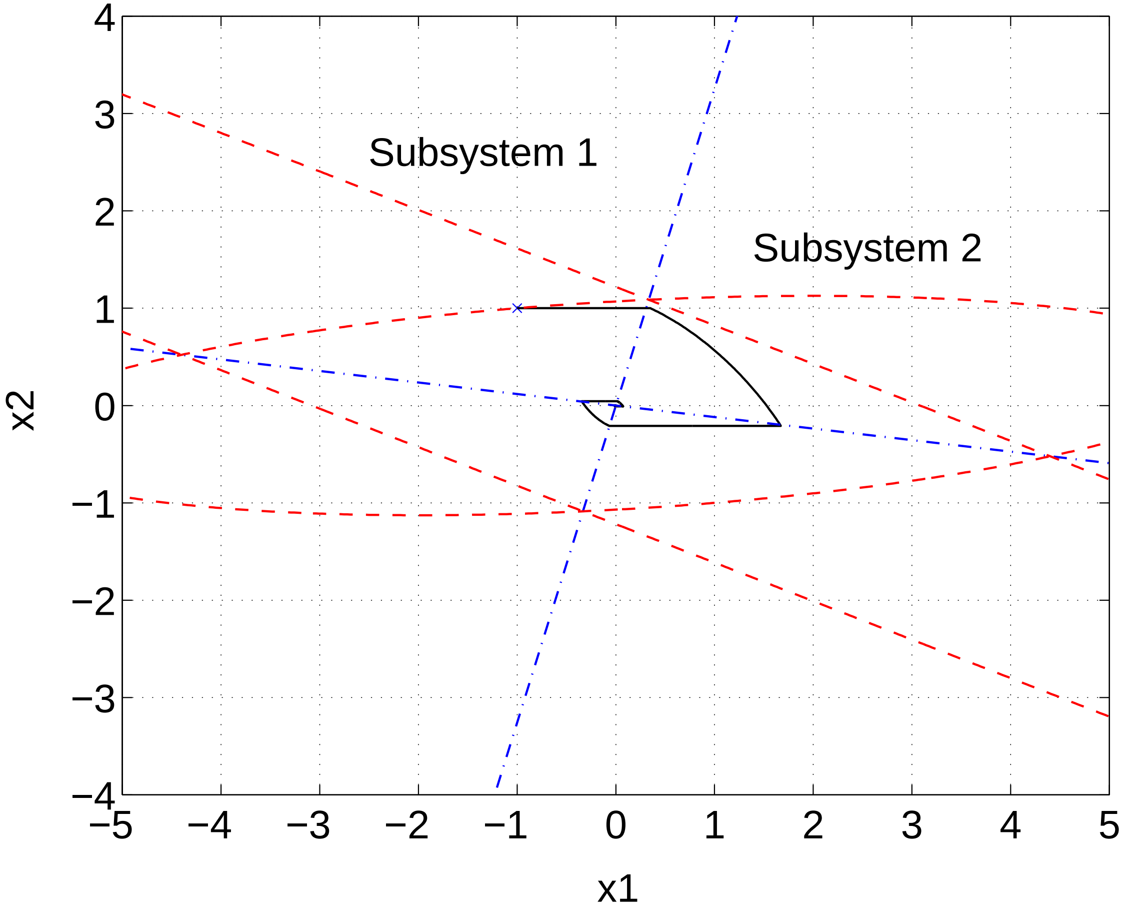
- Used the branch & bound algorithm.
- $r = 0.05$, $\gamma = 0.1$, and $(P_{10}, P_{20}) = (I, 2I)$
- 2000 random states (uniform pdf) for empirical probability calculation
- With $M = 4$ (i.e., 2^4 pairs), after 20 steps and 11 updates,

$$(P_1, P_2) = \left(\begin{bmatrix} 0.0475 & -0.0841 \\ -0.0841 & 1.50 \end{bmatrix}, \begin{bmatrix} 0.180 & 0.456 \\ 0.456 & 1.15 \end{bmatrix} \right).$$

- The equivalent BMI condition holds.

Example 1 (cont'd)

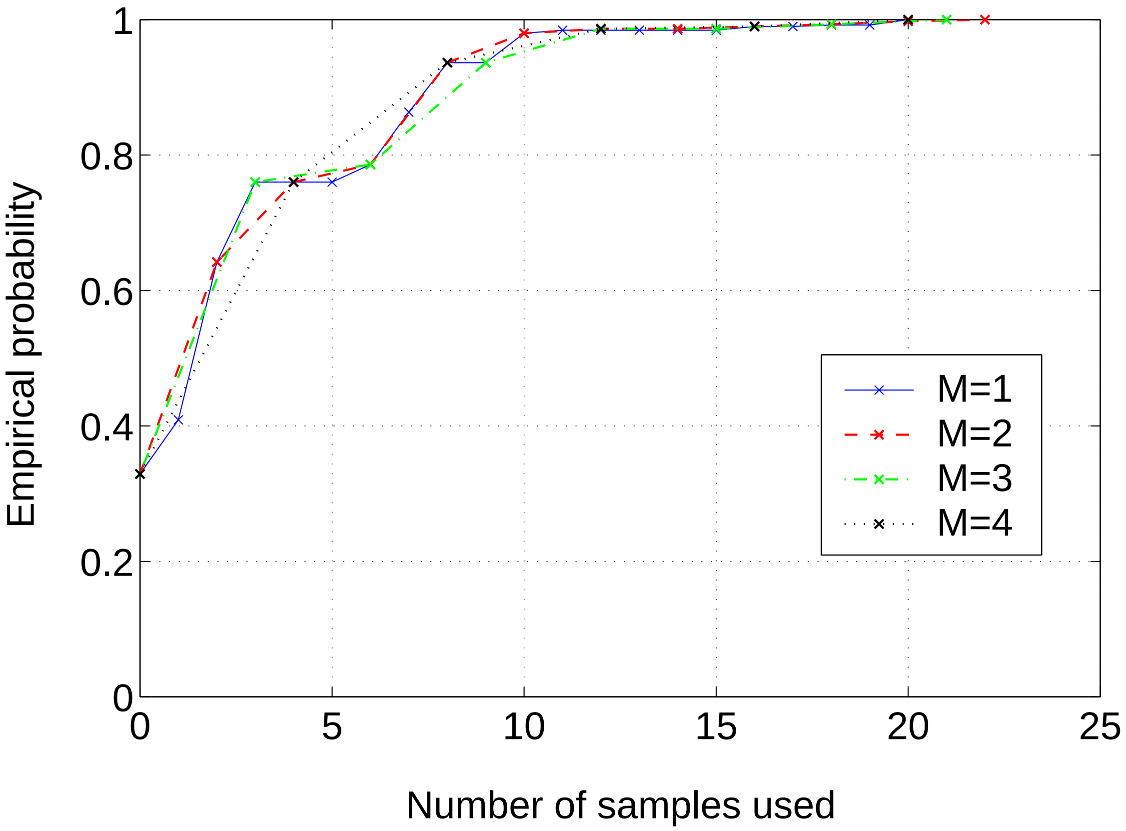
A trajectory



Example 1 (cont'd)

Convergence rates for $M = 1, 2, 3$, and 4

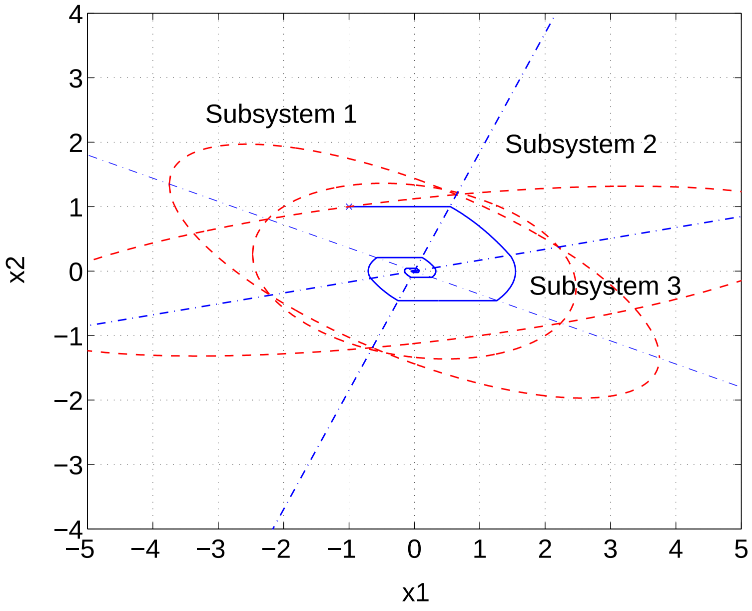
Empirical probability of (P_1, P_2) to be in $\mathcal{P}_s$



Example 2: 2-dim. 3 subsystems

$$A_1 = \begin{bmatrix} 0 & 10 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1.5 & 2 \\ -2 & -0.5 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0.01 & 4 \\ -1 & 0.1 \end{bmatrix}$$

- With $M = 3$, a solution was obtained after 39 steps and 17 updates.
- BMI condition: Only sufficient and does not hold.



Example 3: 3-dim. 2 subsystems

Higher order examples:

Chattering may occur. Then, stability is not guaranteed.

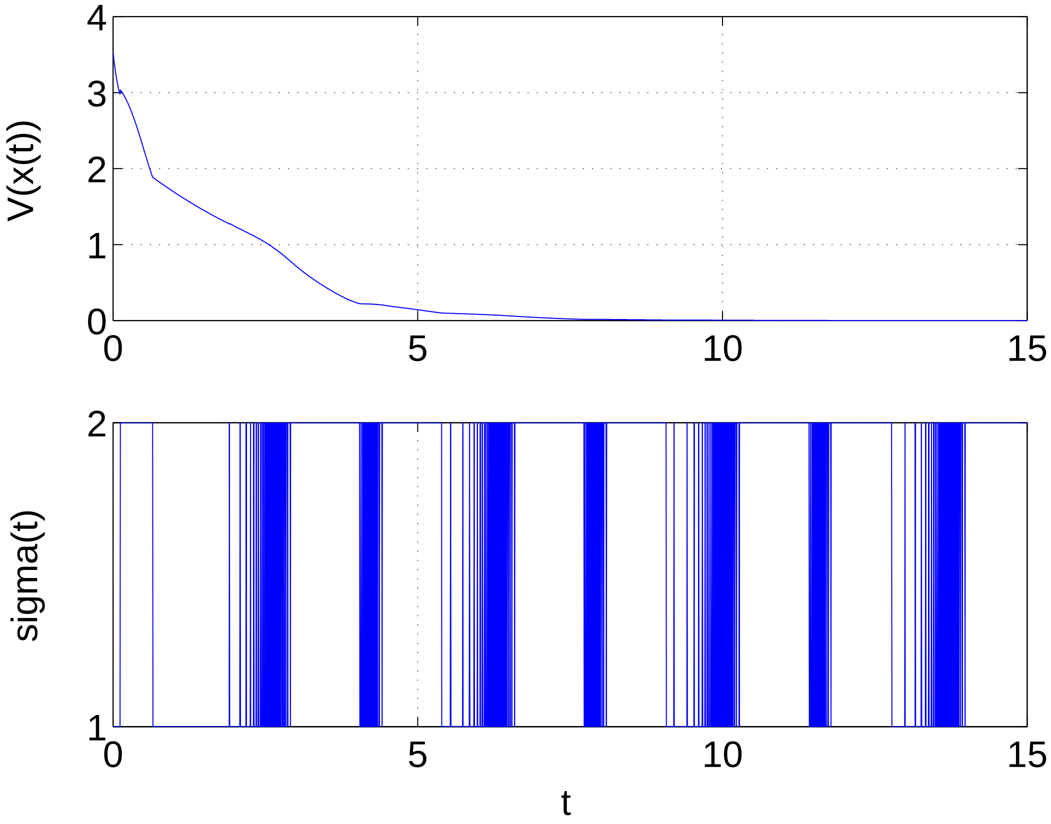
$$A_1 = \begin{bmatrix} 0 & 10 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1.5 & 2 & 0 \\ -2 & -0.5 & 0 \\ 0 & 0 & -0.5 \end{bmatrix}$$

- Added another dimension to A_1 and A_2 in Example 1.
- x_3: Decoupled from other states. Stable.
- One solution:

$$(P_1, P_2) = \left(\begin{bmatrix} 0.0475 & -0.0841 & 0 \\ -0.0841 & 1.50 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0.180 & 0.456 & 0 \\ 0.456 & 1.15 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

Example 3 (cont'd)

- With $M = 4$, obtained a solution after 24 steps and 13 updates.



Lyapunov function $V(x(t))$ and switching signal $\sigma(t)$

Conclusion

- Stabilizing switching rule design for switched systems
- Randomized algorithm with probability-one convergence
- Practical version using branch and bound
- Extensions to multi-modal and nonlinear cases
- Future research: Control problems with the special nonconvex structure